V. *The Initial Accelerated Motion of Electrified Systems of Finite Extent, and the Reaction Produced by the Resulting Radiation.*

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Communicated by Sir Joseph Larmor, *Sec.R.S.*

Received December 17, 1909—Read January 27, 1910.

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1. Introductory.—As far as I am aware, the problem of the electrodynamics of accelerated motion of finite bodies has not been seriously attacked. Attention has been mainly directed to the question of very small charged bodies, and even in this case the results are not as exact as one could desire, in view of the important bearing they have on the theory of the constitution of matter.

Among results which have been generally accepted two call for special attention. The first of these is the equation of linear motion of a small charged sphere determined by Lorentz ('Théorie Electromagnétique,' p. 124). In the usual notation this equation is

$$(m+m')\ddot{x}-k\dddot{x} = \mathrm{X},$$

where

$$m' = \tfrac{2}{3}e^2/aC^2 \qquad \text{and} \qquad k = \tfrac{2}{3}e^2/C^3,$$

e being the charge and a the radius of the sphere.

The distribution of the charge on the sphere is not specified, and LORENTZ does not claim any great generality for the equation. We shall find, however, that more refined calculation proves that LORENTZ' equation is an exceedingly good approximation for vibratory motion.

The second result of importance is the rate of radiation from an accelerated charged particle calculated by LARMOR ('Æther and Matter,' p. 227).

The rate is found to be $\frac{2}{3}e^2(\ddot{x})^2/C^3$. The result is based on the Poynting flux over a surface surrounding the particle, and reference to the original calculation shows that certain terms are neglected as small. If the motion is vibratory this is readily seen to be correct, but it has been claimed that the result is true for a uniformly accelerated linear motion. The substitution of the requisite form for the displacement of the particle in this case shows that the terms neglected are as important as those retained, and the result must be modified. It is further important to note that MACDONALD ('Electric Waves,' p. 72) has shown that a term (*nil* for periodic motion) must be added to the Poynting flux in order to give the whole rate of radiation. Since a uniform movement in a circle may be compounded of two vibratory motions there is no reason why LARMOR's result should fail in this case, although the acceleration is uniform.

LARMOR's result has, I think, been applied to the motion of a charged particle, without regard to the condition of validity.

THOMSON gives the equation ('Conduction of Electricity Through Gases,' p. 543)

$$(m+m')\ddot{x} + \tfrac{2}{3}\frac{e^2}{C^3}\frac{(\ddot{x})^2}{\dot{x}} = X,$$

where the term

$$\tfrac{2}{3}\frac{e^2}{C^3}\frac{(\ddot{x})^2}{\dot{x}}$$

is the reaction on account of the radiation

$$\tfrac{2}{3}\frac{e^2}{C^3}(\ddot{x})^2.$$

Against such an equation two criticisms may be made. First, it does not appear where the term $m'\ddot{x}$ comes from, as it ought to arise from the total rate of radiation. Second, for a given velocity it gives two values for $(\ddot{x})$, which may be real or imaginary—a conclusion which seems untenable. The difficulty here presented may be partially removed by consideration of LORENTZ' equation

$$M\ddot{x} - k\dddot{x} = X.$$

Strictly, M and k are functions of the frequency, but with this limitation we obtain as an integral

$$\tfrac{1}{2}\mathrm{M}\dot{x}^2 - k\dot{x}\ddot{x} + k\int \ddot{x}^2\,dt = \int \mathrm{X}\,dx.$$

On average for a periodic motion the term $k\dot{x}\ddot{x}$ disappears, and we get the equation of balance of energy, for the mean value of $k\int x^2\,dt$ is the quantity of radiation as calculated by LARMOR.

It is clear that we cannot reverse this process, and that it breaks down when the motion is not periodic.

A similar objection applies to the more elaborate expansion by SOMMERFELD ('Gött. Nachrichten,' p. 410, 1904) for the reaction in powers of the acceleration.

ABRAHAM ('Electrician,' p. 868, 1904) has given a still more general formula for the reaction. Apart from difficulties as to the distribution of the charge on the particle, his expression does not enable one to determine the important question as to what kind of motion is really possible. Many of the calculations I have seen either ignore the surface conditions or introduce assumptions about rigid electrification which seriously detract from the value of the conclusions.

Experimental work of recent years has naturally directed attention to the problem of the dependence of electrical inertia on the speed. Since the problem of accelerated motion has not hitherto been solved, extensive use has been made of the solution for steady motion. The process of deriving an expression for the electrical inertia from the expression for the energy of the steady motion has given rise to ambiguity of meaning which is inevitable with such a method, and involves a serious fallacy of dynamical reasoning.

It will be generally admitted that if we introduce steady motion values in a proper Lagrangean energy function, and then apply the usual methods, we have no right to expect correct results. This fallacy is shown by the example $y = x - a$, thus $dy/dx = 1$ for all the values of x if we first differentiate, but if we put $x = a$ and then differentiate we get $dy/dx = 0$.

But, apart from this, a fallacy is involved. If the energy function has been derived from a Newtonian system of equations and the kinetic energy involves squares of the velocities, inertia may be defined from the energy function in a variety of ways, each of which gives the same result. Thus, if

$$\text{K.E.} = \mathrm{T} = \tfrac{1}{2}mu^2,$$

we may define mass as

$$\frac{2\mathrm{T}}{u^2}, \quad \frac{1}{u}\frac{d\mathrm{T}}{du}, \quad 2\frac{d}{du}\frac{\mathrm{T}}{u}, \quad \text{or} \quad \frac{d^2\mathrm{T}}{du^2}.$$

We may devise an infinite number of definitions, all of which are consistent as long as $\mathrm{T} = \tfrac{1}{2}mu^2$.

If, however, we find by any process that the kinetic energy involves higher powers

u than u^2, then not only do the former definitions give different results, but also we are not justified in attempting to form any equation of motion at all or draw any conclusion about the inertia.

The method of investigation by means of the Poynting vector of energy flux (ABRAHAM, 'Ann. d. Physik,' 10, 1903), although possessing some elegant features, is open to many objections. In addition to those raised by SOMMERFELD (*loc. cit.*, p. 368) we must add that of MACDONALD already mentioned. Further, the satisfaction of surface conditions become very difficult. The method is very suitable for determining the field due to rigidly electrified systems moving in a prescribed way, but does not reveal the manner in which the prescribed motion is established. The systems to be considered are not rigidly electrified, and our problem is the determination of the motion, and the way in which it is produced, subject to the necessary surface conditions.

To give a definite instance, it will be shown that a uniformly accelerated motion of a charged sphere is established by aid of a rapidly damped harmonic train of waves. Knowing this to be the case, we might use the method to verify the result, but the method itself does not suggest the occurrence of this damped harmonic train. Thus as a means of discovery it lacks an essential element.

Although these considerations had not been definitely formulated when I attacked the problem of accelerated motion, I had a very distinct impression that the Newtonian method of investigation would prove the most effective. The measure of success of the following investigations confirms a growing belief that Newtonian methods give a more direct correspondence with physical phenomena than any other process that has been devised.

2. *Fundamental Equations.*—It is now generally accepted that the electromagnetic equations of the free stagnant æther are unaffected by the motion of electrified bodies. Thus while such motion gives rise to electric and magnetic actions they must conform to the equations for the stagnant æther.

Hence, if X, Y, Z are the components of electric force, α, β, γ are the components of magnetic force, and C the velocity of radiation, the equations referred to a right-handed system of fixed axes are

$$\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\quad \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\quad \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)=\frac{1}{\mathrm{C}}\left(\frac{\partial \mathrm{X}}{\partial t},\quad \frac{\partial \mathrm{Y}}{\partial t},\quad \frac{\partial \mathrm{Z}}{\partial t}\right),$$

$$\left(\frac{\partial \mathrm{Y}}{\partial z}-\frac{\partial \mathrm{Z}}{\partial y},\quad \frac{\partial \mathrm{Z}}{\partial x}-\frac{\partial \mathrm{X}}{\partial z},\quad \frac{\partial \mathrm{X}}{\partial y}-\frac{\partial \mathrm{Y}}{\partial x}\right)=\frac{1}{\mathrm{C}}\left(\frac{\partial\alpha}{\partial t},\quad \frac{\partial\beta}{\partial t},\quad \frac{\partial\gamma}{\partial t}\right),$$

$$\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=0,$$

$$\frac{\partial \mathrm{X}}{\partial x}+\frac{\partial \mathrm{Y}}{\partial y}+\frac{\partial \mathrm{Z}}{\partial z}=0.$$

It is generally more convenient to use the equations for a moving origin. Thus, if the origin moves in the direction of x, so that the displacement from a fixed point at any time is ξ, the equations become

$$\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\ \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\ \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)=\frac{1}{C}\left(\frac{\partial}{\partial t}-\dot{\xi}\frac{\partial}{\partial x}\right)\left(X,\ Y,\ Z\right),$$

$$\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y},\ \frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z},\ \frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=\frac{1}{C}\left(\frac{\partial}{\partial t}-\dot{\xi}\frac{\partial}{\partial x}\right)\left(\alpha,\ \beta,\ \gamma\right),$$

$$\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=0,$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0.$$

These equations are exact.

In the following investigations attention has been mainly directed to perfect conductors, on account of the simplification thereby secured. In later sections it will be shown how insulating bodies may be treated.

It is thus necessary to consider what conditions must be satisfied in and on the conductor itself.

Two possible views may be taken about this question.

In the case of steady motion, THOMSON ('Recent Researches,' p. 18) intrinsically uses the condition that the tangential component of electric force (X, Y, Z) should vanish at the surface of the charged body. Now, the force effective in producing motion of electricity, or the electrodynamic force, is not the electric force (X, Y, Z) but is (X′, Y′, Z′), where

$$(X', Y', Z')=\left(X-\frac{\dot{z}}{C}\beta+\frac{\dot{y}}{C}\gamma,\quad Y-\frac{\dot{x}}{C}\gamma+\frac{\dot{z}}{C}\alpha,\quad Z-\frac{\dot{y}}{C}\alpha+\frac{\dot{x}}{C}\beta\right),$$

$\dot{x}, \dot{y}, \dot{z}$, being the components of velocity of the moving point.

LARMOR ('Æther and Matter,' p. 152) concludes that the tangential component of (X′, Y′, Z′) should vanish at the surface of a conductor in steady motion. Strictly, we have no equations for the interior of a perfect conductor, but we define it as a body incapable of supporting electric stress. It seems to me necessary to make (X, Y, Z) and (α, β, γ) vanish, although from one point of view it might suffice to make (X′, Y′, Z′) vanish throughout the conductor.

We have next to determine how these quantities inside are related to the similar quantities outside the surface. The fundamental equations integrated through a thin shell in the usual way show that the tangential component of (X′, Y′, Z′) is continuous, but the fact that we have no right to assign the fundamental equations or the above form of (X′, Y′, Z′) to the inside of a conductor raises a doubt. If (X′, Y′, Z′) is continuous as regards tangential components, then the tangential component of

(X, Y, Z) is discontinuous because the tangential component of (α, β, γ) is discontinuous and measures the surface current. To avoid this discontinuity of æther strain tangentially, I consider the possibility that the tangential component of (X, Y, Z) is continuous. It follows at once that for a perfect conductor we must have either the condition (1) that the tangential component of (X, Y, Z) vanishes, or the condition (2) that the tangential component of (X′, Y′, Z′) vanishes, just outside the surface.

The motion we propose to consider is not, however, steady, but is supposed to be variable. The condition inside the conductor remains the same, and on the basis of continuity of æther strain tangentially we again get condition (1). The argument by which condition (2) is established for steady motion does not appear to me quite so satisfactory when the motion is variable. An experimental difficulty arises to my mind in this connection. We know that a copper sphere can be set in rotation by a rotating magnetic field, and that the motion of a copper plate is rapidly damped in a magnetic field. If condition (2) holds generally for variable motion, the tangential forces that actually exist in these experiments are not explained. Condition (1), however, provides an explanation, because it gives a tangential component of (X′, Y′, Z′) at the surface.

I cannot claim to have proved condition (1), nor am I convinced of its correctness; and, on the other hand, condition (2) seemed to present difficulties. The position seemed to call for reservation of judgment, and the only course was to work out the cases for both conditions.

The distinction is, of course, immaterial when squares of the velocity are neglected.

In dealing with large velocities the question of LORENTZ' hypothesis, that a body contracts in the direction of the motion in the proportion $(1-v^2/C^2)^{\frac{1}{2}}$, naturally arose.

So far, no dynamical explanation of such a hypothesis has been obtained, and considerable doubt still exists as to whether it is really necessary. In dealing with a varying velocity the hypothesis would clearly introduce complications of a somewhat unsatisfactory nature from a dynamical point of view, if from no other, and I have therefore decided to exclude it from the problem.

The method of investigation to be pursued was suggested by reading LOVE's paper on "The Mode of Decay of Vibratory Motions" ('Proc. Lond. Math. Soc.,' ser. 2, vol. 2). In that paper an exceedingly elegant method of dealing with the vibrations of a fixed electrified spherical conductor is established. The electrification is initially constrained to be proportional to a zonal harmonic, and the constraint is then removed. It appears that rapidly damped harmonic or exponential trains of waves are produced, and equations for the determination of the constants are obtained.

On examination of the expression for an initial zonal distribution of the first order, it appeared that if the sphere carried, in addition, a constant surface charge, then a mechanical force, due to the radiation, was exerted in the direction of the axis of the harmonic, and the force vanished only when the vibrations had subsided. It was thus argued conversely, that if a charged sphere was initially at rest and an accelerating force applied, the initial motion would be attended to a first order of approximation by the production of a damped harmonic train of radiation depending on a zonal harmonic of the first order, and that equations could be formed for the motion of the sphere. This method proved successful ('Roy. Soc. Proc.,' A, vol. 77, p. 260), and was extended to a second approximation. It was next found to be applicable to a disturbance from a state of steady motion with any velocity in a straight line.

3. *Initial Motion of a Charged Conducting Sphere.*—This problem was solved in the paper just referred to, but the accelerating force was supposed to arise from a uniform electric field. In order to obtain the electrical effects of the motion free from a superposed electric field, it is convenient to suppose that the accelerating force is of a purely mechanical nature. As the procedure is almost identical with that given ('Roy. Soc. Proc.,' *loc. cit.*), it is sufficient to note that if ξ is the displacement in the direction of x the primary equations are

$$a^2\chi''(\mathrm{C}t-a)+a\chi'(\mathrm{C}t-a)+\chi(\mathrm{C}t-a)-e\xi/\mathrm{C}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$m\ddot{\xi}+\tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi''(\mathrm{C}t-a)=\mathrm{F}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where F is the mechanical accelerating force.

The initial conditions are $\chi=\chi'=0$ when $r=\mathrm{C}t+a$ and $\xi=\dot{\xi}=0$ when $t=0$.

If we write m' for $\frac{2}{3}e^2/a\mathrm{C}^2$ the solutions are

$$\chi(\mathrm{C}t-r)=\mathrm{A}e^{-(\mathrm{C}t-r+a)/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{1/2}\frac{(\mathrm{C}t-r+a)}{2a}+\epsilon\right\}$$
$$+\tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\,\mathrm{C}^3}\left\{(\mathrm{C}t-r+a)^2-\frac{2am}{(m+m')}(\mathrm{C}t-r+a)-\frac{2a^2mm'}{(m+m')^2}\right\},$$

$$\xi=-\tfrac{2}{3}\frac{e\mathrm{A}}{ma\mathrm{C}}e^{-\mathrm{C}t/2a}\sin\left\{\left(3+\frac{4m'}{m}\right)^{1/2}\frac{\mathrm{C}t}{2a}+\epsilon\right\}$$
$$+\tfrac{1}{2}\frac{\mathrm{F}}{(m+m')}\left\{t^2+\frac{2m'}{(m+m')}\frac{at}{\mathrm{C}}+\frac{2m'^2}{(m+m')^2}\frac{a^2}{\mathrm{C}^2}\right\},$$

where

$$\mathrm{A}\sin\epsilon=\frac{e\mathrm{F}a^2mm'}{\mathrm{C}^3(m+m')^3}$$

and

$$\left(3+\frac{4m'}{m}\right)^{1/2}\mathrm{A}\cos\epsilon=\frac{e\mathrm{F}a^2(2m+3m')\,m}{\mathrm{C}^3(m+m')^3}.$$

It may be observed that the initial displacement expressed by the damped harmonic part is equal and opposite to that expressed by the non-periodic portion. After one complete vibration the amplitude of the vibratory part falls to

$$e^{-2\pi/\left(3+\frac{4m'}{m}\right)^{1/2}}.$$

of its initial value, and thus the vibratory motion may practically become insignificant before the equations become invalid. Since the decay is exponential this can be secured for moderate values of Ct/a while the whole displacement can be made small by making F small.

In these circumstances the displacement of the sphere is adequately represented by

$$\xi = \tfrac{1}{2}\frac{\mathrm{F}}{(m+m')}\left\{t^2+\frac{2m'at}{(m+m')\,\mathrm{C}}+\frac{2m'^2a^2}{(m+m')^2\mathrm{C}^2}\right\}$$

while

$$\chi(\mathrm{C}t-r) = \tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\,\mathrm{C}^3}\left\{(\mathrm{C}t-r+a)^2-\frac{2am}{(m+m')}(\mathrm{C}t-r+a)-\frac{2a^2mm'}{(m+m')^2}\right\}$$

throughout a certain region.

Within this region the state of the field is given by

$$(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z}) = \frac{e}{r^3}(x,\ y,\ z)+(1,\ 0,\ 0)\,\tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\,\mathrm{C}^2}\frac{(a^2-r^2)}{r^3}$$
$$-(x^2,\ xy,\ xz)\,\tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\,\mathrm{C}^2}\frac{(3a^2+r^2)}{r^5},$$

$$(\alpha,\ \beta,\ \gamma) = \frac{1}{r^3}(0,\ -z,\ y)\frac{e\mathrm{F}}{(m+m')\,\mathrm{C}}\left\{t+\frac{m'a}{(m+m')\,\mathrm{C}}\right\}.$$

At greater distances the damped harmonic train would have to be retained.

Hence a constant surface density is established, given by

$$\sigma = \frac{1}{4\pi}\left\{\frac{e}{a^2}-\frac{2e\mathrm{F}}{(m+m')a\,\mathrm{C}^2}\mathrm{P}_1\right\}.\text{*}$$

We further find that

$$\dot{\xi} = \frac{\mathrm{F}}{(m+m')}\left(t+\frac{m'}{m+m'}\cdot\frac{a}{\mathrm{C}}\right),$$

$$\ddot{\xi} = \frac{\mathrm{F}}{(m+m')}.$$

Hence the sphere arrives at the point ξ as if the equation of motion had been

$$(m+m')\,\ddot{\xi} = \mathrm{F}$$

with an initial velocity $\dfrac{\mathrm{F}m'a}{(m+m')^2\,\mathrm{C}}$ and an initial displacement $\dfrac{\mathrm{F}m'^2a^2}{(m+m')^3\,\mathrm{C}^2}$. We have thus shown that a uniformly accelerated motion is possible, and that the reaction of the medium is $m'\ddot{\xi}$.

If we introduce a new variable, ϕ, defined by $e\phi/\mathrm{C} = \chi-e\xi/\mathrm{C}$ the equations (1) and (2) become the equations of motion of a system in which the kinetic energy is

$$\mathrm{T} = \tfrac{1}{2}m'(\dot{\phi}+\dot{\xi})^2+\tfrac{1}{2}m\dot{\xi}^2+\tfrac{1}{4}m'(\xi\dot{\phi}-\phi\dot{\xi})\,\mathrm{C}/a,$$

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* In 'Roy. Soc. Proc.,' *loc. cit.*, p. 265, line 5, write $m/(m+m')$ for $\frac{1}{2}(2m+m')/(m+m')$.

the dissipation function

$$D = \tfrac{1}{2}m'\dot{\phi}\,(\dot{\phi}+\dot{\xi})\,C/a,$$

and the potential energy

$$V = \tfrac{1}{2}m'\phi^2C^2/a^2.$$

It is important to note that a dissipation function is required, and also that a gyrostatic term has to be introduced in the kinetic energy. It gives a hint as to the pure dynamics of electro-magnetism. That such a term should occur might be expected from the fundamental equations of the theory, but in the general energy methods of treating electrodynamics I can find no explicit reference to such a term, nor do I see that it could be obtained by other than a Newtonian method. Having obtained the term, it would no doubt be easy to show that it is included in the energy function, but this illustrates exactly MACDONALD'S contention ('Electric Waves,' chap. I.), that the modified Lagrangean function itself cannot be used to determine the concealed motions.

The equation $(m+m')\,\ddot{\xi} = F$, which we have seen may rapidly come to obtain, may be held to suggest absence of radiation. This, however, is not really the case. We have already remarked that the true solution, while consistent with this equation, gives an apparent initial velocity and initial displacement, originally connected with the damped harmonic train.

The rate of dissipation 2 D is found to be

$$-\frac{m'F^2}{(m+m')^2}\left(t-\frac{m}{m+m'}\frac{a}{C}\right),$$

which shows that when $t > \dfrac{m}{m+m'}\cdot\dfrac{a}{C}$, the effective part of the dissipation is really negative, suggesting that energy is being supplied to the system. Initially, to avoid this, we should thus have to include the vibratory part.

If at a time t_1 the accelerating force ceases, the sphere settles down to a steady state with a constant velocity. This is accomplished by the production of a new damped harmonic train.

We may carry out the solution as before, and when the new damped harmonic train becomes negligible, we find that

$$\xi = \frac{Ft_1(t-t_1)}{(m+m')}+\frac{1}{2}\frac{Ft_1^2}{(m+m')}+\frac{am'Ft_1}{C(m+m')^2},$$

and the field is

$$(X, Y, Z) = \frac{e}{r^3}(x, y, z),$$

$$(\alpha, \beta, \gamma) = \frac{e}{r^3}(0, -z, y)\frac{Ft_1}{(m+m')}.$$

Thus the velocity finally established is $Ft_1/(m+m')$, which is the velocity acquired by the system having inertia $(m+m')$ acted on by the force F for a time t_1. Thus the apparent contribution to the initial velocity produced by the first vibrations is exactly destroyed by the second vibrations. In a similar way the apparent contribution to the initial displacement is destroyed.

The contribution $\frac{1}{2}\frac{\mathrm{F}t_1^2}{(m+m')}$ is the displacement due to F acting on $(m+m')$ for a time t_1. The contribution $\frac{am'\mathrm{F}t_1}{\mathrm{C}\,(m+m')^2}$, on account of the apparent initial velocity, could not be expected to disappear.

There is no loss of energy since the velocity established is $\frac{\mathrm{F}t_1}{(m+m')}$, and the energy of the system is $\frac{1}{2}\frac{\mathrm{F}^2t_1^2}{(m+m')}$ or $\mathrm{F}\times\frac{1}{2}\frac{\mathrm{F}t_1^2}{(m+m')}$, and is thus the work done by force F acting on $(m+m')$ for a time t_1. The dissipation function is now found to vanish.

The result then shows that the initial motion is attended by the production of a damped harmonic train. On account of the rapidity of damping, a uniformly accelerated motion soon becomes possible.

The existence of a gyrostatic term in the kinetic energy has been revealed, and also the existence of a dissipation function.

The production of waves (Röntgen radiation) by the sudden creation or destruction of a velocity has been already shown by THOMSON ('Conduction of Electricity through Gases,' p. 538). Our investigation shows that the establishment of a constant velocity is really attended by the production of two rapidly damped harmonic trains, which of course combine if the time of action of the force is sufficiently short. The frequency of the waves is $\left(3+4\frac{m'}{m}\right)^{1/2}\frac{\mathrm{C}}{4\pi a}$ and the modulus of decay $\mathrm{C}/2a$.

4. *Second Order approximation.*—When the former expressions for the field are carried out to squares and products of ξ and $\chi\,(\mathrm{C}t-r)$, it appears that the motion is modified by the production of damped harmonic waves depending on a second order zonal harmonic.

We therefore introduce a new function $\chi_2\,(\mathrm{C}t-r)$ associated with a second order zonal harmonic supposed to be small of the second order, while ξ and $\chi_1\,(\mathrm{C}t-r)$ are small of the first order. (X′, Y′, Z′) now differs from (X, Y, Z) by terms of the second order.

We can readily show that this will introduce terms of not less than the third order in the equation of motion of the sphere when condition (1) is used.

Thus the surface density is given by $4\pi\sigma = \mathrm{N}$, where N is the normal component of electric force. Since the tangential component must vanish at the surface, the force due to radiation reaction is

$$= \tfrac{1}{2}\int \sigma \mathrm{N} \mathrm{P}_1\, d\mathrm{S},$$

$$= 2\pi \int \sigma^2 \mathrm{P}_1\, d\mathrm{S}.$$

Thus, if

$$\sigma = \sigma_0 + \sigma_1 P_1 + \sigma_2 P_2,$$

where

$$\sigma_1 \text{ is of the first order,}$$

and

$$\sigma_2 \text{ is of the second order,}$$

the force becomes

$$\begin{aligned} &= 4\pi^2 a^2 \int_{-1}^{+1} P_1 (\sigma_0 + \sigma_1 P_1 + \sigma_2 P_2)^2 \, d\mu, \\ &= 4\pi^2 a^2 \left(\tfrac{4}{3}\sigma_0\sigma_1 + \tfrac{8}{15}\sigma_1\sigma_2\right). \end{aligned}$$

This result can readily be extended, and it appears that only products of successive σ's occur.

I had a special reason for wishing to examine the effect of a uniform field of electric force F in this case, so that the new meaning of F must be remembered in comparing results with those of the preceding section.

The forms to be assumed for the field in accordance with the fundamental equations up to the second order are

$$\begin{aligned} (X, Y, Z) = {} & \frac{e}{r^3}(x, y, z) + \frac{C}{r^5}(x^2, xy, xz)\left(r^2\chi_1'' + 3r\chi_1' + 3\chi_1 - 3e\xi/C\right) \\ & + (1, 0, 0)\left\{F - \frac{C}{r^3}\left(r^2\chi_1'' + r\chi_1' + \chi_1 - e\xi/C\right)\right\} \\ & + \frac{e\xi^2}{r^5}(x, y, z)\left\{1 + \tfrac{5}{2}(3x^2 - r^2)/r^2\right\} \\ & + \frac{C\xi}{r^5}(x, y, z)\left\{\left(r^2\chi_1'' + 3r\chi_1' + 3\chi_1\right) - \frac{x^2}{r^2}\left(r^3\chi_1''' + 6r^2\chi_1'' + 15r\chi_1' + 15\chi_1\right)\right\} \\ & + \frac{C}{r^5}(x, y, z)\left\{\left(r^2\chi_2'' + 3r\chi_2' + 3\chi_2\right) - \frac{x^2}{r^2}\left(r^3\chi_2''' + 6r^2\chi_2'' + 15r\chi_2' + 15\chi_2\right)\right\} \\ & + \frac{C}{r^5}(x, 0, 0)\left\{\left(r^3\chi_2''' + 3r^2\chi_2' + 6r\chi_2' + 6\chi_2\right) + \xi\left(r^3\chi_1''' + 3r^2\chi_1'' + 6r\chi_1' + 6\chi_1\right) - \frac{3e\xi^2}{C}\right\}. \end{aligned}$$

Hence, in order that the tangential component of (X, Y, Z) should vanish at the surface, we must have for $r = a$

$$\begin{aligned} a^2\chi_1'' + a\chi_1' + \chi_1 - e\xi/C &= a^3 F/C, \\ a^3\chi_2''' + 3a^2\chi_2'' + 6a\chi_2' + 6\chi_2 &= \frac{3e\xi^2}{C} - \xi\left(a^3\chi_1''' + 3a^2\chi_1'' + 6a\chi_1' + 6\chi_1\right). \end{aligned}$$

Hence we find that the surface density of electricity is given by σ, where

$$4\pi\sigma = \frac{e}{a^2} + \frac{C \,.\, P_1}{a^3}\left(\frac{3a^3 F}{C} - 2a^2\chi_1''\right) + \frac{C \,.\, P_2}{a^4}\left\{a^3\chi_2''' + a^2\chi_2'' + \xi\left(a^3\chi_1''' + a^2\chi_1''\right)\right\},$$

while the equation of motion of the sphere is

$$m\ddot{\xi} = \tfrac{1}{3}\frac{eC}{a^3}\left(\frac{3a^3 F}{C} - 2a^2\chi_1''\right) + \tfrac{2}{15}\frac{C^2}{a^5}\left(\frac{3a^3 F}{C} - 2a^2\chi_1''\right)\left\{a^3\chi_2''' + a^2\chi_2'' + \xi\left(a^3\chi_1''' + a^2\chi_1''\right)\right\}.$$

There is theoretically an additional reaction due to the magnetic force acting on the surface current. It depends on the fourth power of the velocity, and has therefore been neglected.

Vibrations then of the first and second order arise, but since these are rapidly damped we may fix attention on the motion possible when these have become negligible.

A trial of the assumption $\xi = \frac{1}{2}k_1t^2$, and determination of the ensuing forms for χ_1 and χ_2 from the conditional equations, shows that in the equation of motion we leave unbalanced terms of the third order in k_1 which depend on the time.

A uniformly accelerated motion is thus impossible with a constant force, but we may destroy the terms of the third order in the equation of motion by assuming for ξ the form

$$\xi = \tfrac{1}{2}k_1t^2+\tfrac{1}{3}k_2t^3+\tfrac{1}{4}k_3t^4,$$

where k_2 and k_3 are at least of the third order. For the satisfaction of this condition we require

$$k_1 = \frac{e\mathrm{F}}{(m+m')}\left\{1+\tfrac{2}{5}\frac{a^3\mathrm{F}^2m\,(3mm'^2-m^3-m'^3)}{\mathrm{C}^2\,(m+m')^5}\right\},$$

$$k_2 = -\tfrac{3}{5}\frac{mm'^2}{(m+m')^5}\,\frac{a^2e\mathrm{F}^3}{\mathrm{C}},$$

$$k_3 = -\tfrac{1}{5}\frac{m'mae\mathrm{F}^3}{(m+m')^4}.$$

In these expressions higher powers of F have been neglected.

The values of (X′, Y′, Z′) are obtained by adding to the expressions for (X, Y, Z) the vector $-(0,\, y,\, z)\,(r\chi_1''+\chi_1')\,\dot{\xi}/r^3$.

Hence, using condition (2), the equations that hold at $r = a$ are now

$$a^2\chi_1''+a\chi_1'+\chi_1-e\xi/\mathrm{C} = a^3\mathrm{F}/\mathrm{C},$$

$$a^3\chi_2'''+3a^2\chi_2''+6a\chi_2'+6\chi_2 = 3e\xi^2/\mathrm{C}-\xi\,(a^3\chi_1'''+3a^2\chi_1''+6a\chi_1'+6\chi_1)-\dot{\xi}\,(a\chi_1''+\chi_1')\,a^2/\mathrm{C},$$

while the surface density is given by σ, where

$$4\pi\sigma = \frac{e}{a^2}+\frac{\mathrm{CP}_1}{a^3}\left(\frac{3a^3\mathrm{F}}{\mathrm{C}}-2a^2\chi_1''\right)+\frac{\mathrm{CP}_2}{a^4}\{a^3\chi_2'''+a^2\chi_2''+\xi(a^3\chi_1'''+a^2\chi_1'')+\dot{\xi}(a\chi_1''+\chi_1')a^2/\mathrm{C}\}.$$

The equation of motion is

$$m\ddot{\xi} = e\left(\mathrm{F}-\tfrac{2}{3}\frac{\mathrm{C}}{a}\chi_1''\right)+\tfrac{2}{5}\frac{\mathrm{C}}{a^2}\left(\mathrm{F}-\tfrac{2}{3}\frac{\mathrm{C}}{a}\chi''\right)\{a^3\chi_2'''+a^2\chi_2''+\xi(a^3\chi_1'''+a^2\chi_1'')+\dot{\xi}(a\chi_1''+\chi_1')a^2/\mathrm{C}\}.$$

Proceeding as before, by assuming

$$\xi = \tfrac{1}{2}k_1t^2+\tfrac{1}{3}k_2t^3+\tfrac{1}{4}k_3t^4,$$

we find, up to terms of the third order in F, that $k_2 = k_3 = 0$ and

$$k_1 = \frac{e\mathrm{F}}{m+m'}\left\{1+\tfrac{1}{5}\frac{a^3\mathrm{F}^2}{\mathrm{C}^2}\,\frac{m\,(3m'-2m)}{(m+m')^3}\right\}.$$

Thus, using condition (1), a uniformly accelerated motion under a constant force is not possible, but since the deviation depends on the third power of F, it is clear that a high degree of accuracy can be claimed for the results of the preceding section. Using condition (2), a uniform acceleration is possible, including the third power of F.

The constant part of the acceleration is modified in a way which depends on the relative magnitudes of m and m'. In each case the effect may be to increase or diminish the electric inertia by the existence of the field. The result differs from that obtained by HEAVISIDE ('Nature,' April 19, 1906), and afterwards by SEARLE ('Nature,' June 28, 1906), who find that the field always increases the electric inertia. The argument is based on the energy of the steady state, and I have already shown that no legitimate inference as to inertia can be drawn from this.

It is noteworthy that if the field F was of the same strength as that produced by the charged sphere at the surface, viz., e/a^2, the term a^3F^2/C^2 would equal e^2/aC^2 or $\frac{3}{2}m'$.

If the approximation is valid for such a field, the effective modification of the electric inertia would thus be very considerable.

This conclusion is of very great importance in experiments on Becquerel or Kathode rays, where we must suppose that a large number of charged particles are moving very close together. It seems impossible to estimate how much effect would be produced, but that some modification of the effective inertia would result from the mutual field of the charged particles is beyond doubt.

5. *Initial Motion of a Charged Conducting Sphere moving with any Speed after Longitudinal Acceleration is imposed.*—The problem of the steady linear motion of a charged sphere using condition (1) has been solved by THOMSON ('Recent Researches,' p. 17).

We now proceed to investigate the effect of an accelerating force in the direction of the existing motion.

The general equations for the field in the æther referred to a fixed origin are those in Section 2.

If we refer the system to a moving origin, for which the displacement parallel to x at any time is $kCt+f(t)$, where k is a constant, the equations become

$$\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\ \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\ \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)=\frac{1}{C}\left\{\frac{\partial}{\partial t}-(kC+f't)\frac{\partial}{\partial x}\right\}(X,\ Y,\ Z),$$

$$\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y},\ \frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z},\ \frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=\frac{1}{C}\left\{\frac{\partial}{\partial t}-(kC+f't)\frac{\partial}{\partial x}\right\}(\alpha,\ \beta,\ \gamma),$$

$$\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=0,$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0.$$

When $f(t) = 0$ we get the equations for the case of uniform motion, and as a solution of these equations independent of time, we get

$$\alpha_0 = 0, \quad \beta_0 = k\frac{\partial\psi_0}{\partial z}, \quad \gamma_0 = -k\frac{\partial\psi_0}{\partial y},$$

$$X_0 = -(1-k^2)\frac{\partial\psi_0}{\partial x}, \quad Y_0 = -\frac{\partial\psi_0}{\partial y}, \quad Z_0 = -\frac{\partial\psi_0}{\partial z},$$

where ψ_0 is a solution of

$$(1-k^2)\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2} + \frac{\partial^2\psi_0}{\partial z^2} = 0.$$

In order to pass to a disturbed motion, we may assume $f(t)$ to be small, and that the electric and magnetic forces differ from the steady values by X, Y, Z, α, β, γ, which are small of the same order as $f(t)$.

Hence, neglecting squares and products of small quantities, the fundamental equations are

$$\left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}, \ \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}, \ \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) = \frac{1}{C}\left(\frac{\partial}{\partial t} - kC\frac{\partial}{\partial x}\right)(X, Y, Z) - \frac{f'(t)}{C}\frac{\partial}{\partial x}(X_0, Y_0, Z_0),$$

$$\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \ \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \ \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = \frac{1}{C}\left(\frac{\partial}{\partial t} - kC\frac{\partial}{\partial x}\right)(\alpha, \beta, \gamma) - \frac{f'(t)}{C}\frac{\partial}{\partial x}(\alpha_0, \beta_0, \gamma_0),$$

$$\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z} = 0,$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

When condition (1) is used it is convenient to assume the system

$$\alpha = 0,$$

$$\beta = \frac{1}{C}\frac{\partial^2\phi}{\partial t\,\partial z} - k\frac{\partial^2\phi}{\partial x\,\partial z} + kf\frac{\partial^2\psi_0}{\partial x\,\partial z},$$

$$\gamma = -\frac{1}{C}\frac{\partial^2\phi}{\partial t\,\partial y} + k\frac{\partial^2\phi}{\partial x\,\partial y} - kf\frac{\partial^2\psi_0}{\partial x\,\partial y},$$

$$X = -(1-k^2)f\frac{\partial^2\psi_0}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2},$$

$$Y = -f\frac{\partial^2\psi_0}{\partial x\,\partial y} + \frac{\partial^2\phi}{\partial x\,\partial y},$$

$$Z = -f\frac{\partial^2\psi_0}{\partial x\,\partial z} + \frac{\partial^2\phi}{\partial x\,\partial z},$$

where ϕ is a solution of

$$\left(\frac{\partial}{\partial t} - kC\frac{\partial}{\partial x}\right)^2\phi = C^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

and ϕ is supposed to be small of the same order as f, but there is no restriction to a small value of k.

In the particular case of a charged conducting sphere, we have to satisfy the condition that the tangential component of electric force should vanish at $r = a$, and that the surface density of electricity is given by $\frac{1}{4\pi}$ (normal component of electric force).

The appropriate form for ψ_0 is

$$\psi_0 = (1-k^2)^{-1/2} e/\rho,$$

where

$$\rho^2 = \frac{x^2}{1-k^2} + y^2 + z^2$$

and e is the total charge on the sphere, which is not uniformly distributed.

The corresponding simplest form for ϕ is

$$\phi = \rho^{-1}\chi\{Ct - \lambda x - \rho/(1-k^2)^{1/2}\}$$

where $\lambda = k/(1-k^2)$ and χ is an arbitrary function.

Thus the contributions to electric force are

$$\begin{aligned}(X, Y, Z) = {} & \frac{1}{\rho^3}(-1, 0, 0)\left[\chi - \frac{ef}{(1-k^2)^{1/2}} + \left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi' + \frac{\rho}{(1-k^2)^{1/2}}\left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi''\right] \\ & + \frac{k}{(1-k^2)\rho^3}(x, y, z)\left\{\chi' + \frac{\rho}{(1-k^2)^{1/2}}\chi''\right\} \\ & + \frac{x}{(1-k^2)\rho^5}(x, y, z)\left[3\left\{\chi - \frac{ef}{(1-k^2)^{1/2}}\right\} + \frac{3\rho}{(1-k^2)^{1/2}}\chi' + \frac{\rho^2}{(1-k^2)}\chi''\right].\end{aligned}$$

Now, the finite terms due to the steady state are already chosen to secure the vanishing of the tangential component of (X_0, Y_0, Z_0), hence the tangential component of (X, Y, Z) will vanish at $r = a$, provided

$$\chi - \frac{ef}{(1-k^2)^{1/2}} + \left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi' + \frac{\rho}{(1-k^2)^{1/2}}\left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi'' = 0$$

when $r = a$, for all values of t.

As in the simpler case in Section 3, damped harmonic vibrations will arise and rapidly become negligible. We therefore proceed to consider the motion established when the vibrations have subsided.

Let

$$f = \tfrac{1}{2}\ddot{f}_0 t^2 \text{ where } \ddot{f}_0 \text{ is constant,}$$

then, if

$$\chi = \frac{1}{2}\frac{e\ddot{f}_0}{(1-k^2)^{1/2}C^2}\left[\left\{Ct - \frac{\rho}{(1-k^2)^{1/2}} - \lambda x\right\}^2 - \frac{a^2}{(1-k^2)}\right]$$

the tangential condition is satisfied identically.

Hence the total components of electric force at $r = a$ are

$$(\mathrm{X}_0+\mathrm{X},\ \mathrm{Y}_0+\mathrm{Y},\ \mathrm{Z}_0+\mathrm{Z}) = \frac{e}{(1-k^2)^{1/2}\rho^3}(x, y, z),$$

$$+\frac{k}{(1-k^2)\rho^3}(x, y, z)\left\{1-\frac{3x^2}{(1-k^2)\rho^2}\right\}\chi'(\mathrm{C}t),$$

$$+\frac{x}{(1-k^2)\rho^3}(x, y, z)\left\{1-\frac{3a^2}{(1-k^2)\rho^2}\right\}\chi''(\mathrm{C}t).$$

The surface density of electricity is given by σ where

$$4\pi\sigma = \frac{ea}{(1-k^2)^{1/2}\rho^3}+\frac{ka}{(1-k^2)\rho^3}\left\{1-\frac{3x^2}{(1-k^2)\rho^2}\right\}\chi'(\mathrm{C}t)+\frac{ax}{(1-k^2)\rho^3}\left\{1-\frac{3a^2}{(1-k^2)\rho^2}\right\}\chi''(\mathrm{C}t).$$

It may be verified that the terms in $\chi'(\mathrm{C}t)$ and $\chi''(\mathrm{C}t)$ contribute zero to the total electric charge as is required.

Since there is a surface current in addition to the convection current due to transference of the sphere, the mechanical reaction on the sphere is

$$= \tfrac{1}{2}\int\{\sigma(\mathrm{X}_0+\mathrm{X})-w(\beta_0+\beta)+v(\gamma_0+\gamma)\}\,d\mathrm{S},$$

where u, v, w are the components of current determined by the surface discontinuity of magnetic force and the integration is taken over the sphere.

Neglecting squares of small quantities the value is

$$\tfrac{1}{2}\frac{e\chi''(\mathrm{C}t)}{a(1-k^2)^{3/2}}\int_{-1}^{+1}\left[\frac{x^2}{\left(1+\dfrac{k^2x^2}{1-k^2}\right)^3}\left\{1-\frac{3}{(1-k^2)\left(1+\dfrac{k^2x^2}{1-k^2}\right)}\right\}+\frac{k^2}{2}\frac{x^2(1-x^2)}{\left(1+\dfrac{k^2x^2}{1-k^2}\right)^3}\right]dx.$$

The evaluation of the integral, which is somewhat tedious, gives for the reaction the value

$$\tfrac{1}{16}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{4-13k^2+6k^4}{k^2(1-k^2)}-\frac{(4-5k^2+4k^4)}{k^3(1-k^2)^{3/2}}\sin^{-1}k\right\}.$$

The equation of motion of the sphere under a force F is thus

$$m\ddot{f}_0 = \mathrm{F}+\tfrac{1}{16}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{4-13k^2+6k^4}{k^2(1-k^2)}-\frac{4-5k^2+4k^4}{k^3(1-k^2)^{3/2}}\sin^{-1}k\right\}.$$

We thus prove that, to the given order of approximation, a uniformly accelerated motion is possible as soon as the vibrations subside, and conclude that the initial electric inertia for longitudinal acceleration is

$$\tfrac{1}{16}\frac{e^2}{a\mathrm{C}^2}\left\{\frac{(4-5k^2+4k^4)}{k^3(1-k^2)^{3/2}}\sin^{-1}k-\frac{(4-13k^2+6k^4)}{k^2(1-k^2)}\right\}.$$

The limiting value of this expression when $k = 0$ is found to be $\frac{2}{3} e^2/aC^2$, thus agreeing with the result in Section 3.

When condition (2) is used the method of procedure is very similar to that already used.

The components of electrodynamic force are

$$X'_0 = -(1-k^2)\frac{\partial\psi_0}{\partial x}, \quad Y'_0 = -(1-k^2)\frac{\partial\psi_0}{\partial y}, \quad Z'_0 = -(1-k^2)\frac{\partial\psi_0}{\partial z}.$$

The condition that the tangential component of (X'_0, Y'_0, Z'_0) should vanish at $r = a$ is equivalent to the condition that ψ_0 should be constant at $r = a$ The solution of this problem is given by MACDONALD ('Electric Waves,' p. 172) in the form

$$\psi_0 = \frac{e}{ka} \log \coth \tfrac{1}{2}\eta,$$

where

$$\frac{x^2}{\cosh^2 \eta} + \frac{(1-k^2)(y^2+z^2)}{\sinh^2 \eta} = k^2a^2.$$

It appears that the surface density of electricity is uniform and equal to $e/4\pi a^2$.

In proceeding to the disturbed state it was found convenient to modify slightly the expressions formerly used.

With the same restrictions as before the total field in the disturbed state is given by

$$\alpha = 0,$$

$$\beta = k\frac{\partial\psi_0}{\partial z} + \frac{1}{C}\frac{\partial^2\phi}{\partial t\,\partial z} - k\frac{\partial^2\phi}{\partial x\,\partial z} + kf\frac{\partial^2\psi_0}{\partial x\,\partial z} - k\frac{\partial\chi}{\partial z},$$

$$\gamma = -k\frac{\partial\psi_0}{\partial y} - \frac{1}{C}\frac{\partial^2\phi}{\partial t\,\partial y} + k\frac{\partial^2\phi}{\partial x\,\partial y} - kf\frac{\partial^2\psi_0}{\partial x\,\partial y} - k\frac{\partial\chi}{\partial y},$$

$$X = -(1-k^2)\frac{\partial\psi_0}{\partial x} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} + (1-k^2)\frac{\partial\chi}{\partial x},$$

$$Y = -\frac{\partial\psi_0}{\partial y} - f\frac{\partial^2\psi_0}{\partial x\,\partial y} + \frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial\chi}{\partial y},$$

$$Z = -\frac{\partial\psi_0}{\partial z} - f\frac{\partial^2\psi_0}{\partial x\,\partial z} + \frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial\chi}{\partial z}.$$

Hence

$$X' = -(1-k^2)\frac{\partial\psi_0}{\partial x} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} + (1-k^2)\frac{\partial\chi}{\partial x},$$

$$Y' = -(1-k^2)\frac{\partial\psi_0}{\partial y} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x\,\partial y} + (1-k^2)\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{k}{C}\frac{\partial^2\phi}{\partial t\,\partial y} + \frac{kf'}{C}\frac{\partial\psi_0}{\partial y} + (1-k^2)\frac{\partial\chi}{\partial y},$$

$$Z' = -(1-k^2)\frac{\partial\psi_0}{\partial z} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x\,\partial z} + (1-k^2)\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{k}{C}\frac{\partial^2\phi}{\partial t\,\partial z} + \frac{kf'}{C}\frac{\partial\psi_0}{\partial z} + (1-k^2)\frac{\partial\chi}{\partial z}.$$

In these χ, which is independent of t, satisfies

$$(1-k^2)\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}+\frac{\partial^2\chi}{\partial z^2}=0,$$

and ϕ satisfies

$$\left(\frac{\partial}{\partial t}-k\mathrm{C}\frac{\partial}{\partial x}\right)^2\phi=\mathrm{C}^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right),$$

while χ and ϕ are both small of the same order as f.

Vibrations may be supposed to arise and subside rapidly. It is clearly not possible to determine these in a simple form since the question corresponds now to vibrations on a spheroid at rest.

Such vibrations do not lend themselves to analysis in the same way as vibrations on a sphere, and only approximate treatment has been found possible.

Our object, however, is the determination of the motion after vibrations have subsided, and with this limitation it has been found possible to complete the solution.

The determination of ϕ and χ to satisfy the surface conditions proved an exceedingly difficult problem. The process was in great measure tentative, and does not possess much intrinsic interest. When obtained the solutions can be verified in a straightforward although slightly tedious manner.

If

$$f=\tfrac{1}{2}\ddot{f}_0t^2,\text{ where }\ddot{f}_0\text{ is constant,}$$

then

$$\phi=\left\{\mathrm{C}t-\frac{kx}{(1-k^2)}\right\}^2\phi_1+\left\{\mathrm{C}t-\frac{kx}{(1-k^2)}\right\}\phi_2+\phi_3,$$

where

$$\phi_1=\tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\psi_0,\qquad \psi_0=\frac{e}{ka}\log\coth\tfrac{1}{2}\eta,$$

$$\phi_2=\tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\cdot\frac{2x}{k(1-k^2)}\left\{\psi_0-\frac{e}{ka\cosh\eta}\right\},$$

$$\phi_3=\tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\left[\frac{1}{3(1-k^2)}\left\{\frac{x^2}{(1-k^2)}+y^2+z^2+\frac{(3-k^2)a^2}{(1-k^2)}\right\}\psi_0+\tfrac{2}{3}\frac{eka}{(1-k^2)^2}\cosh\eta\right],$$

and

$$\chi=\tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\left\{\frac{1}{(1-k^2)^2}(\tfrac{4}{3}-\mathrm{B})\,x\left(\psi_0-\frac{e}{ka\cosh\eta}\right)\right\},$$

where

$$\mathrm{B}=1\Big/\left(1-\frac{1}{2k}\log\frac{1+k}{1-k}\right).$$

These expressions substituted in the equations satisfy the condition that $(\mathrm{X}', \mathrm{Y}', \mathrm{Z}')$ should be entirely radial at $r=a$.

The components of electric force at $r = a$ are

$$\mathrm{X} = \frac{e\,(1-k^2)\,x}{a\,(a^2-k^2x^2)} - \frac{2\ddot{f}_0 ek\mathrm{C}tx\,(a^2-x^2)}{\mathrm{C}^2a\,(a^2-k^2x^2)^2} + \frac{\ddot{f}_0 ex^2}{\mathrm{C}^2a\,(a^2-k^2x^2)}\left(1+\frac{k^2\mathrm{B}}{1-k^2}\right),$$

$$\mathrm{Y} = \frac{ey}{a\,(a^2-k^2x^2)} + \frac{2\ddot{f}_0 ek\mathrm{C}tx^2y}{\mathrm{C}^2a\,(a^2-k^2x^2)^2} + \frac{\ddot{f}_0 exy}{\mathrm{C}^2a\,(a^2-k^2x^2)\,(1-k^2)}\left(1+\frac{k^2\mathrm{B}}{1-k^2}\right),$$

$$\mathrm{Z} = \frac{ez}{a\,(a^2-k^2x^2)} + \frac{2\ddot{f}_0 ek\mathrm{C}tx^2z}{\mathrm{C}^2a\,(a^2-k^2x^2)^2} + \frac{\ddot{f}_0 exz}{\mathrm{C}^2a\,(a^2-k^2x^2)\,(1-k^2)}\left(1+\frac{k^2\mathrm{B}}{1-k^2}\right).$$

Thus the surface density of electricity is given by

$$4\pi\sigma = \frac{e}{a^2} + \frac{\ddot{f}_0 ex}{\mathrm{C}^2a\,(1-k^2)}\left(1+\frac{k^2\mathrm{B}}{1-k^2}\right).$$

There is thus a redistribution on the sphere while the total charge remains unaltered.

The mechanical reaction on the sphere in the direction of x is $\frac{1}{2}\int\sigma\mathrm{X}'\,d\mathrm{S}$, since in this case there is no surface current, and $\mathrm{X}' = \mathrm{X}$.

The term in $\ddot{f}_0 t$ vanishes on integration, and hence, neglecting squares of $\ddot{f}_0$, the value is

$$\frac{1}{2}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{1}{k^3}\log\frac{(1+k)}{(1-k)} - \frac{2}{k^2\,(1-k^2)}\right\}.$$

Hence the equation of motion is

$$m\ddot{f}_0 = \mathrm{F} + \frac{1}{2}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{1}{k^3}\log\frac{(1+k)}{(1-k)} - \frac{2}{k^2\,(1-k^2)}\right\}.$$

Thus a uniformly accelerated motion is possible, and the initial electric inertia for longitudinal acceleration is

$$\frac{1}{2}\frac{e^2}{a\mathrm{C}^2}\left\{\frac{2}{k^2\,(1-k^2)} - \frac{1}{k^3}\log\frac{(1+k)}{(1-k)}\right\}.$$

This result is the same as that of ABRAHAM for a rigidly electrified sphere. The investigation shows, however, that a redistribution of the charge takes place.

The limiting value of the expression for $k = 0$ is $\frac{2}{3}e^2/a\mathrm{C}^2$.

6. *Initial Motion of a Charged Conducting Sphere moving with any Speed after Transverse Acceleration is imposed.*—The sphere being in steady motion with velocity $k\mathrm{C}$ parallel to x, we now suppose the accelerating force to act at right angles to the original direction of motion.

Thus we now take $f(t)$ as a small displacement parallel to the direction of y.

Referred to the centre of the sphere as a moving origin the steady state is given as before by

$$\alpha_0 = 0, \quad \beta_0 = k\frac{\partial\psi_0}{\partial z}, \quad \gamma_0 = -k\frac{\partial\psi_0}{\partial y},$$

$$X_0 = -(1-k^2)\frac{\partial\psi_0}{\partial x}, \quad Y_0 = -\frac{\partial\psi_0}{\partial y}, \quad Z_0 = -\frac{\partial\psi_0}{\partial z}.$$

Proceeding, as before, to a first order approximation, we find that the field due to the disturbance, using condition (1), is conveniently expressed by the system

$$\begin{aligned}
\alpha &= -\frac{1}{C}\frac{\partial^2\phi}{\partial t\,\partial z},\\
\beta &= kf\frac{\partial^2\psi_0}{\partial y\,\partial z} - k\frac{\partial^2\phi}{\partial z\,\partial y},\\
\gamma &= \frac{1}{C}\frac{\partial^2\phi}{\partial tx} - kf\frac{\partial^2\psi_0}{\partial y^2} + k\frac{\partial^2\phi}{\partial y^2},\\
X &= -(1-k^2)f\frac{\partial^2\psi_0}{\partial x\,\partial y} + (1-k^2)\frac{\partial^2\phi}{\partial y\,\partial x} + \frac{k}{C}\frac{\partial^2\phi}{\partial t\,\partial y},\\
Y &= -f\frac{\partial^2\psi_0}{\partial y^2} - (1-k^2)\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial z^2} - \frac{k}{C}\frac{\partial^2\phi}{\partial t\,\partial x},\\
Z &= -f\frac{\partial^2\psi_0}{\partial y\,\partial z} + \frac{\partial^2\phi}{\partial y\,\partial z},
\end{aligned}$$

where ϕ is a solution of

$$\left(\frac{\partial}{\partial t} - kC\frac{\partial}{\partial x}\right)^2\phi = C^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right),$$

supposed small of the same order as f.

We assume the same forms as before

$$\psi_0 = (1-k^2)^{-1/2}e/\rho,$$

$$\phi = \rho^{-1}\chi\left\{Ct - \frac{\rho}{(1-k^2)^{1/2}} - \lambda x\right\}.$$

The contributions to electric force are

$$\begin{aligned}
(X, Y, Z) = \frac{1}{\rho^3}(0, -1, 0)&\left[\left\{\chi - \frac{ef}{(1-k^2)^{1/2}}\right\} + \left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi' + \frac{\rho}{(1-k^2)^{1/2}}\left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi''\right]\\
&+ \frac{y}{\rho^5}(x, y, z)\left\{3\chi - \frac{3ef}{(1-k^2)^{1/2}} + \frac{3\rho}{(1-k^2)^{1/2}}\chi' + \frac{\rho^2}{(1-k^2)}\chi''\right\}.
\end{aligned}$$

The tangential component of electric force vanishes at $r = a$ if

$$\chi - \frac{ef}{(1-k^2)^{1/2}} + \left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi' + \frac{\rho}{(1-k^2)^{1/2}}\left\{\frac{\rho}{(1-k^2)^{1/2}} + \lambda x\right\}\chi'' = 0,$$

when $r = a$ for all values of t.

If, as before, we consider the state when the vibrations have subsided, we find that the tangential condition is satisfied by assuming

$$f = \tfrac{1}{2}\ddot{f}_0 t^2,$$

$$\chi = \tfrac{1}{2}\frac{e\ddot{f}_0}{(1-k^2)^{1/2}\mathrm{C}^2}\left[\left\{\mathrm{C}t - \frac{\rho}{(1-k^2)^{1/2}} - \lambda x\right\}^2 - \frac{a^2}{(1-k^2)}\right].$$

The total components of electric force at the surface are

$$\begin{aligned}(\mathrm{X}_0+\mathrm{X},\ \mathrm{Y}_0+\mathrm{Y},\ \mathrm{Z}_0+\mathrm{Z}) &= \frac{e}{(1-k^2)^{1/2}\rho^3}(x,\ y,\ z)\\ &\quad - \frac{3kxy}{(1-k^2)\,\rho^5}(x,\ y,\ z)\,\chi'(\mathrm{C}t)\\ &\quad + \frac{y}{(1-k^2)\,\rho^3}(x,\ y,\ z)\left(1-\frac{3a^2}{\rho^2}\right)\chi''(\mathrm{C}t).\end{aligned}$$

The surface density of the electricity σ is given by

$$4\pi\sigma = \frac{ea}{(1-k^2)^{1/2}\rho^3} - \frac{3kaxy}{(1-k^2)\,\rho^5}\chi'(\mathrm{C}t) + \frac{ay}{(1-k^2)\,\rho^3}\left(1-\frac{3a^2}{\rho^2}\right)\chi''(\mathrm{C}t).$$

The terms in $\chi'(\mathrm{C}t)$ and $\chi''(\mathrm{C}t)$ contribute zero to the total charge of the sphere.

The mechanical reaction on the sphere in the direction of y is

$$= \tfrac{1}{2}\int\{\sigma(\mathrm{Y}_0+\mathrm{Y}) - u(\gamma_0+\gamma) + w(\alpha_0+\alpha)\}\,d\mathrm{S},$$

where the components of current are determined as before from the surface discontinuity of magnetic force.

Reducing the expression and neglecting squares and products of small quantities, we get for the mechanical reaction

$$-\tfrac{1}{8}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{(4k^2-1)}{k^3(1-k^2)^{1/2}}\sin^{-1}k + \frac{1+2k^2}{k^2}\right\}.$$

Thus the equation of motion under a force F is

$$m\ddot{f}_0 = \mathrm{F} - \tfrac{1}{8}\frac{e^2\ddot{f}_0}{a\mathrm{C}^2}\left\{\frac{(4k^2-1)}{k^3(1-k^2)^{1/2}}\sin^{-1}k + \frac{1+2k^2}{k^2}\right\}.$$

Hence, to this order of approximation, a uniformly accelerated transverse motion is possible when the vibrations have subsided.

The initial electric inertia for a transverse acceleration is thus

$$\tfrac{1}{8}\frac{e^2}{a\mathrm{C}^2}\left\{\frac{(4k^2-1)}{k^3(1-k^2)^{1/2}}\sin^{-1}k + \frac{1+2k^2}{k^2}\right\}.$$

The expression, when converted in notation, is identical with that obtained by J. J. Thomson ('Recent Researches,' p. 21).

The limiting value for $k = 0$ is found to be as before, $\frac{2}{3}e^2/a\mathrm{C}^2$.

Using condition (2), the forms to be assumed for the total field are

$$\alpha = -\frac{1}{\mathrm{C}}\frac{\partial^2\phi}{\partial t\,\partial z},$$

$$\beta = k\frac{\partial\psi_0}{\partial z} + kf\frac{\partial^2\psi_0}{\partial y\,\partial z} - k\frac{\partial^2\phi}{\partial y\,\partial z} - k\frac{\partial\chi}{\partial z},$$

$$\gamma = -k\frac{\partial\psi_0}{\partial y} + \frac{1}{\mathrm{C}}\frac{\partial^2\phi}{\partial t\,\partial z} - kf\frac{\partial^2\psi_0}{\partial y^2} + k\frac{\partial^2\phi}{\partial y^2} + k\frac{\partial\chi}{\partial y},$$

$$\mathrm{X} = -(1-k^2)\frac{\partial\psi_0}{\partial x} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x\,\partial y} + (1-k^2)\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{k}{\mathrm{C}}\frac{\partial^2\phi}{\partial t\,\partial y} + (1-k^2)\frac{\partial\chi}{\partial x},$$

$$\mathrm{Y} = -\frac{\partial\psi_0}{\partial y} - f\frac{\partial^2\psi_0}{\partial y^2} - (1-k^2)\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial z^2} - \frac{k}{\mathrm{C}}\frac{\partial^2\phi}{\partial t\,\partial x} + \frac{\partial\chi}{\partial y},$$

$$\mathrm{Z} = -\frac{\partial\psi_0}{\partial z} - f\frac{\partial^2\psi_0}{\partial y\,\partial z} + \frac{\partial^2\phi}{\partial y\,\partial z} + \frac{\partial\chi}{\partial z}.$$

Hence

$$\mathrm{X}' = -(1-k^2)\frac{\partial\psi_0}{\partial x} - (1-k^2)f\frac{\partial^2\psi_0}{\partial x\,\partial y} + (1-k^2)\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{k}{\mathrm{C}}\frac{\partial^2\phi}{\partial t\,\partial y} - \frac{kf'}{\mathrm{C}}\frac{\partial\psi_0}{\partial y} + (1-k^2)\frac{\partial\chi}{\partial x},$$

$$\mathrm{Y}' = -(1-k^2)\frac{\partial\psi_0}{\partial y} - (1-k^2)f\frac{\partial^2\psi_0}{\partial y^2} + (1-k^2)\frac{\partial^2\phi}{\partial y^2} - \frac{1}{\mathrm{C}^2}\frac{\partial^2\phi}{\partial t^2} + (1-k^2)\frac{\partial\chi}{\partial y},$$

$$\mathrm{Z}' = -(1-k^2)\frac{\partial\psi_0}{\partial z} - (1-k^2)f\frac{\partial^2\psi_0}{\partial y\,\partial z} + (1-k^2)\frac{\partial^2\phi}{\partial y\,\partial z} + (1-k^2)\frac{\partial\chi}{\partial z}.$$

We obtain as a solution when the vibrations have subsided

$$f = \tfrac{1}{2}\ddot{f}_0 t^2,$$

$$\phi = \left\{\mathrm{C}t - \frac{kx}{(1-k^2)}\right\}^2\phi_1 + \left\{\mathrm{C}t - \frac{kx}{(1-k^2)}\right\}\phi_2 + \phi_3,$$

where

$$\phi_1 = \tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\psi_0, \qquad \psi_0 = \frac{e}{ka}\log\coth\tfrac{1}{2}\eta,$$

$$\phi_2 = \tfrac{1}{2}\frac{\ddot{f}_0 2x}{\mathrm{C}^2 k(1-k^2)}\left\{\psi_0 - \frac{e}{ka\cosh\eta}\right\},$$

$$\phi_3 = \tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\left[\frac{1}{3(1-k^2)}\left\{\frac{x^2}{1-k^2} + y^2 + z^2 + \frac{(3-k^2)a^2}{(1-k^2)}\right\}\psi_0 + \tfrac{2}{3}\frac{eka\cosh\eta}{(1-k^2)^2}\right],$$

and

$$\chi = \tfrac{1}{2}\frac{\ddot{f}_0}{\mathrm{C}^2}\left\{\frac{1}{(1-k^2)}\left(\tfrac{4}{3}-\mathrm{B}\right)y\left(\psi_0 - \frac{e}{ka}\frac{\sinh\eta}{\cosh^2\eta}\right)\right\},$$

where

$$\mathrm{B} = 2\Big/\left\{1 - \frac{(1-k^2)}{2k}\log\frac{(1+k)}{(1-k)}\right\}.$$

These forms give at $r = a$

$$(\mathrm{X}', \mathrm{Y}', \mathrm{Z}') = \left\{\frac{e\,(1-k^2)}{a\,(a^2-k^2x^2)} + \frac{2\,\ddot{f}_0 ek\mathrm{C}t\,(1-k^2)\,xy}{\mathrm{C}^2a\,(a^2-k^2x^2)^2} + \frac{\ddot{f}_0 e\,(1+k^2-k^2\mathrm{B})\,y}{\mathrm{C}^2a\,(1-k^2)\,(a^2-k^2x^2)}\right\}(x, y, z),$$

thus securing the condition that (X′, Y′, Z′) should be radial at the surface.

Further, at $r = a$ we have

$$\mathrm{X} = \frac{e\,(1-k^2)\,x}{a\,(a^2-k^2x^2)} + \frac{2\,\ddot{f}_0 ek\mathrm{C}t}{\mathrm{C}^2a}\left\{\frac{(1-k^2)\,x^2y}{(a^2-k^2x^2)^2} - \tfrac{1}{2}\frac{y}{(a^2-k^2x^2)}\right\} + \frac{\ddot{f}_0 exy\,(1+k^2-k^2\mathrm{B})}{\mathrm{C}^2a\,(a^2-k^2x^2)\,(1-k^2)},$$

$$\mathrm{Y} = \frac{ey}{a\,(a^2-k^2x^2)} + \frac{2\,\ddot{f}_0 ek\mathrm{C}t}{\mathrm{C}^2a}\left\{\frac{xy^2}{(a^2-k^2x^2)^2} - \tfrac{1}{2}\frac{x}{(a^2-k^2x^2)}\right\}$$
$$+ \frac{\ddot{f}_0 ey^2\,(1+k^2-k^2\mathrm{B})}{\mathrm{C}^2a\,(a^2-k^2x^2)\,(1-k^2)^2} - \frac{\ddot{f}_0 e}{\mathrm{C}^2a}\left\{1 - \frac{1}{2k}\log\frac{(1+k)}{(1-k)}\right\},$$

$$\mathrm{Z} = \frac{ez}{a\,(a^2-k^2x^2)} + \frac{2\,\ddot{f}_0 ek\mathrm{C}txyz}{\mathrm{C}^2a\,(a^2-k^2x^2)^2} + \frac{\ddot{f}_0 ey\,(1+k^2-k^2\mathrm{B})}{\mathrm{C}^2a\,(a^2-k^2x^2)\,(1-k^2)^2}.$$

Hence the surface density of electricity is given by

$$4\pi\sigma = \frac{e}{a^2} - \frac{\ddot{f}_0 ey}{\mathrm{C}^2a^2}\left\{1 - \frac{1}{2k}\log\frac{(1+k)}{(1-k)}\right\} + \frac{\ddot{f}_0 ey\,(1+k^2-k^2\mathrm{B})}{\mathrm{C}^2a^2\,(1-k^2)^2}.$$

Thus a redistribution of the charge takes place while the total charge is unaltered. The mechanical reaction in the direction of y is $\tfrac{1}{2}\int\sigma\mathrm{Y}'\,d\mathrm{S}$ since there is no surface current. Neglecting squares of $\ddot{f}_0$, we obtain on reduction the value

$$\frac{\ddot{f}_0 e^2}{a\mathrm{C}^2}\left[\tfrac{1}{2}\left\{\frac{1}{k^2} - \frac{(1+k^2)}{2k^3}\log\frac{(1+k)}{(1-k)}\right\} - \tfrac{1}{4}\left\{1 - \frac{1}{2k}\log\frac{(1+k)}{(1-k)}\right\}\left\{\frac{1}{k^2} - \frac{(1-k^2)}{2k^3}\log\frac{(1+k)}{(1-k)}\right\}\right].$$

As in former cases, a uniformly accelerated motion is found to be possible, and the initial electric inertia for a transverse acceleration is

$$\tfrac{1}{2}\frac{e^2}{a\mathrm{C}^2}\left[\left\{\frac{(1+k^2)}{2k^3}\log\frac{(1+k)}{(1-k)} - \frac{1}{k^2}\right\} + \tfrac{1}{2}\left\{\frac{1}{2k}\log\frac{(1+k)}{(1-k)} - 1\right\}\left\{\frac{(1-k^2)}{2k^3}\log\frac{(1+k)}{(1-k)} - \frac{1}{k^2}\right\}\right].$$

This result differs from ABRAHAM's formula in so far as it contains the product term. The limiting value of the expression for $k = 0$ is $\tfrac{2}{3}\,e^2/a\mathrm{C}^2$.

7. *Comparison with Experiment.*—In the preceding sections we have considered the acceleration to be produced by a purely mechanical force. It is perhaps almost directly obvious that if the force is due to a uniform electric field F, no change of electric inertia is produced when F^3 and higher terms are neglected, as we have merely to superpose on the former solutions a uniform field with the appropriate induced electrification on a body moving uniformly. Initially, of course, this state is produced by the aid of a rapidly damped harmonic train. As a matter of fact the problems already solved were first worked out for an electric field which was afterwards annulled with a view to making clear how much of the induced electrification was due to the accelerated motion itself.

The results established are true without limitation as to the size of the sphere, but, as will appear, it is only for velocities comparable with that of radiation that the possibility of experimental discrimination can arise. We are thus at present limited to experiments on negative electrons, and among researches on the deflexion of such particles KAUFMANN's investigations on Becquerel rays rank first in historical order ('Gött. Nachrichten,' 1903, Heft 3; 'Ann. d. Physik,' vol. 19, p. 487, 1906).

We propose to examine three expressions for the transverse electric inertia.

With condition (1) we have

$$m' \frac{3}{16k^2}\left\{\frac{4k^2-1}{k(1-k^2)^{1/2}}\sin^{-1}k+1+2k^2\right\} \quad . \quad . \quad \left\{\begin{array}{c}\text{(J. J. THOMSON)}\\ \text{or}\\ \text{(Present No. 1).}\end{array}\right.$$

With condition (2) we have

$$m' \frac{3}{4k^2}\left\{\frac{1+k^2}{2k}\log\frac{1+k}{1-k}-1\right\} \quad . \quad . \quad . \quad . \quad \text{(M. ABRAHAM)},$$

$$m'\left[\frac{3}{4k^2}\left\{\frac{1+k^2}{2k}\log\frac{1+k}{1-k}-1\right\}+\frac{3}{8k^2}\left\{\frac{1}{2k}\log\frac{1+k}{1-k}-1\right\}\left\{\frac{1-k^2}{2k}\log\frac{1+k}{1-k}-1\right\}\right] \quad \text{(Present No. 2).}$$

In these expressions m' equals $\frac{2}{3}e^2/aC^2$, and k is the ratio of the velocity of the sphere to that of radiation. They all agree in giving the value m' when $k = 0$.

The expressions of THOMSON and ABRAHAM are derived from considerations of the steady state. The ambiguity of interpretation from consideration of a steady state has already been commented on in Section 1, and has been definitely admitted by POINCARÉ. Thus it is not surprising that the expressions differ, although it so happens that we have obtained expressions which agree with THOMSON's result for transverse motion and ABRAHAM's result for longitudinal motion.

The following table gives the numerical values of the co-efficient of m' at various speeds according to the three expressions :—

k.	THOMSON, Present No. 1.	ABRAHAM.	Present No. 2.
·70	1·327	1·295	1·228
·71	1·344	1·308	1·238
·72	1·361	1·322	1·248
·73	1·380	1·337	1·259
·74	1·400	1·353	1·271
·75	1·421	1·369	1·284
·76	1·443	1·387	1·297
·77	1·467	1·405	1·310
·78	1·493	1·424	1·325
·79	1·521	1·445	1·340
·80	1·551	1·468	1·357
·81	1·583	1·491	1·375
·82	1·619	1·516	1·393
·83	1·658	1·543	1·413
·84	1·700	1·573	1·435
·85	1·747	1·604	1·458
·86	1·800	1·639	1·483
·87	1·858	1·677	1·511
·88	1·924	1·718	1·541
·89	2·000	1·764	1·575
·90	2·085	1·816	1·613
·91	2·188	1·874	1·655
·92	2·306	1·940	1·703
·93	2·455	2·016	1·759
·94	2·637	2·107	1·825
·95	2·874	2·217	1·904
·96	3·195	2·356	2·005
·97	3·669	2·540	2·138
·98	4·469	2·808	2·332
·99	6·284	3·286	2·679
Nearly 1	$\frac{9\pi}{32(1-k^2)^{1/2}}$	$\frac{3}{4}\log\frac{(1+k)}{(1-k)}$	$\frac{9}{16}\log\frac{(1+k)}{(1-k)}$

The values are calculated to the nearest unit in the third decimal place, and are, I believe, correct. They have been checked by a professional calculator.

The numbers show that discrimination between ABRAHAM'S formula and No. 2 is a somewhat delicate matter, and would require experiments of a high order of accuracy.

If we assume that the mass of the negative particle in Becquerel rays is $m+m'f(k)$ or μ, where $f(k)$ stands for any of the co-efficients in the three formulæ, we may apply the calculation to KAUFMANN'S observations.

In KAUFMANN'S first paper we have the relations

$$k = \cdot 4175 \frac{\mathrm{F}}{\mathrm{C}.\mathrm{H}} \frac{z'}{y'} = \mathrm{K}_1 \frac{z'}{y'},$$

$$\frac{e}{\mathrm{C}\mu} = 3\cdot 59 \times 10^7 kz',$$

where F is the strength of the electric field, H the strength of the magnetic field, z' is the magnetic, and y' the electric deflexion.

The constant K_1 is directly determined by the conditions of the experiment, and since it is free from any theory as to the way in which μ depends on the value of k, this appears the most satisfactory way in which to proceed. KAUFMANN, however, adopting ABRAHAM'S formula and the view that the whole mass is electric, proceeds to determine the constants which will best fit the experimental curve z', y'. This does not appear to me to be strictly logical, since it gives a bias in favour of the theory adopted. The procedure has been ably criticised by PLANCK ('Phys. Zeit,' 1906, p. 753), and I think we must agree with him in standing by the determination of the constant which is independent of any theory. Unfortunately in the first paper there are not sufficient data to calculate K_1, but we may accept the value ·257 for plate No. 19, which is stated to be in good agreement with the value as reckoned from the conditions of experiment. It ought to be specially favourable to the theory adopted by KAUFMANN.

Plate No. 19 has been selected as the best, according to KAUFMANN, and two readings omitted as clearly subject to some casual error of observation. The values of k are first calculated and then the values of $\frac{1}{kz'}$. These ought to be proportional to

$$\mathrm{A}+\mathrm{B}f(k).$$

The values are then combined in pairs to give three values of B thus,

$$\mathrm{B} = \frac{(4)-(1)}{f(k_4)-f(k_1)}, \qquad \mathrm{B} = \frac{(5)-(2)}{f(k_5)-f(k_2)}, \qquad \mathrm{B} = \frac{(6)-(3)}{f(k_6)-f(k_3)}.$$

These ought to give the same values for B. The mean is taken and used to calculate A. This is theoretically the best mode of combining the observations.

z'.	y'.	k.	$\frac{1}{kz'}$.	Differences.	$f(k)$, Thomson, No. 1.	Differences.	B.	A.
·247	·0678	·936	4·325		2·559			·82
·3435	·1019	·866	3·361		1·834			·85
·391	·1219	·824	3·103		1·635			·86
·437	·1420	·786	2·911	1·414	1·510	1·049	1·35	·84
·4825	·1660	·747	2·774	·587	1·414	·420	1·39	·84
·5265	·1916	·706	2·690	·413	1·337	·298	1·38	·86
							1·37	·84
					Abraham.			
					2·068			·15
					1·661			·01
					1·527			·02
					1·437	·631	2·24	·01
					1·364	·297	1·97	·02
					1·303	·224	1·84	·06
							2·02	·04
					Present No. 2.			
					1·797			− ·58
					1·500			− ·73
					1·401			− ·72
					1·334	·463	3·05	− ·73
					1·280	·220	2·67	− ·72
					1·234	·167	2·47	− ·68
							2·73	− ·69

The superiority of Thomson's formula No. 1 in giving constant values for B and A is at once apparent. Abraham's formula gives distinctly increasing values of B as k increases, and thus does not give a large enough dependence of mass on speed, even if the mass is assumed to be wholly electric. No. 2 gives the same disagreement in the values of B and a negative value to the real mass, a result quite inadmissible.

Selecting No. 1, we obtain

$$\frac{e/\mathrm{C}}{m+m'f(k)} = 1{\cdot}62\times10^7\times\frac{2{\cdot}21}{\{{\cdot}84+1{\cdot}37f(k)\}},$$

and

$$\frac{e/\mathrm{C}}{m+m'} = 1{\cdot}62\times10^7.$$

In his latter experiments, which KAUFMANN considers more accurate, we have

$$k = \frac{\mathrm{F}}{\mathrm{C.H}}\frac{z'}{y'}, \qquad \frac{e}{\mathrm{C}\mu} = \frac{\mathrm{C}}{\mathrm{H}}kz',$$

where

$$\mathrm{F} = 315 \times 10^{10}, \qquad \mathrm{H} = 557{\cdot}1.$$

Hence

$$k = {\cdot}1884\frac{z'}{y'}.$$

As PLANCK has shown, this gives k greater than 1 for KAUFMANN'S smallest values of z' and y', and I have selected four of his readings from Table VII. as falling within a suitable range.

z'.	y'.	k.	$\frac{1}{kz'}$.	Differences.	$f(k)$, THOMSON, No. 1.	Differences.	B.	A.
·2400	·0502	·900	4·629		2·085			1·79
·2890	·0645	·844	4·100		1·718			1·76
·3359	·0811	·780	3·816	·813	1·493	·592	1·37	1·78
·3832	·1001	·721	3·619	·481	1·362	·356	1·35	1·77
							1·36	1·77
					ABRAHAM.			
					1·816			1·09
					1·585			1·01
					1·424	·392	2·07	1·04
					1·323	·262	1·83	1·04
							1·95	1·04
					Present No. 2.			
					1·613			·37
					1·444			·29
					1·325	·288	2·82	·32
					1·249	·195	2·46	·32
							2·64	·32

We again find that THOMSON'S formula No. 1 gives most satisfactory agreement, while ABRAHAM'S formula and No. 2 do not meet the case.

With No. 1 we get

$$\frac{e/\mathrm{C}}{m+m'f(k)} = 1{\cdot}71 \times 10^7 \times \frac{3{\cdot}13}{\{1{\cdot}77+1{\cdot}36f(k)\}}$$

and

$$\frac{e/\mathrm{C}}{m+m'} = 1{\cdot}71 \times 10^7.$$

The difference between the values of $\frac{e/\mathrm{C}}{(m+m')}$ from the two investigations is not perhaps very serious, but the relative magnitudes of m and m' in the two cases is more important. In the first set m appears to be about $\frac{1}{2}m'$, while in the second set m is about equal to m'. Some latitude must, however, be allowed in the value of the constant K_1 in the first set and the ensuing values of k. The function $f(k)$ is extremely sensitive to changes of k, when k is approaching unity, and thus relatively large changes in the calculated ratio of m to m' will be produced. More accurate experiments are necessary to decide this point.

We may, however, claim that formula No. 1 provides a substantial explanation of KAUFMANN's experiments, and assigns to the real mass of the particle a value comparable with the electric mass.

If we wish to hold the view that the mass of an electron is wholly electric we must conclude that the particles in KAUFMANN's experiments are not electrons, but are either charged particles with a real constitutional mass, or electrons which have become attached to gross matter.

The analysis, on the other hand, is distinctly against ABRAHAM's formula and No. 2.

We have no right to conclude that the particles are conductors, as it is still probable that the assumption of perfect insulation would explain the experiments (see Section 9). We may only claim that the assumption of perfect conductivity does not disagree with the facts.

We may, however, fairly argue from the experiments that condition (2) cannot be maintained along with the view that the particles are conductors, while condition (1) with this hypothesis adequately explain the facts.

In forming a judgment of the results of this application of theory to experiment it may be well to recall the concluding paragraph of Section 4.

Since this analysis was made an investigation by BUCHERER (‘Phys. Zeit,’ 1908, p. 755) has appeared. He gives the results of experiments agreeing well with LORENTZ' formula $m'/(1-k^2)^{1/2}$ for a “contracted electron,” but not in agreement with ABRAHAM's formula. I may say that KAUFMANN's experiments also agree excellently with LORENTZ' formula, just as they do with THOMSON's formula, when a proportion of ordinary mass is admitted. The reason is that both formulæ contain an infinity of the form $1/(1-k^2)^{1/2}$. KAUFMANN's results thus seem to me not inferior in accuracy to those of BUCHERER.

LORENTZ' formula is derived from the "quasi-stationary" principle, which may or may not give correct results. But, as was stated in Section 2, I have not yet seen how to apply the method of this paper to a body which alters its dimensions as the velocity alters.

Discussion of BUCHERER'S results by the present method cannot therefore be adequately done. We may note that THOMSON'S formula would not agree with BUCHERER'S numbers as well as does LORENTZ' formula, but would give much better agreement than ABRAHAM'S formula, or the corrected value using condition (2) for an accelerated motion.

8. *Initial Motion of an Insulating Charged Sphere.*—We shall suppose that the sphere, initially at rest, has a uniform surface charge e, that the material has a dielectric ratio K, and that the velocity of radiation in the material is C′, thus

$$\mathrm{K} = \mathrm{C}^2/\mathrm{C}'^2.$$

The equations for the æther outside the sphere remain the same as before, but while the field inside the sphere is initially zero, the motion must give rise to a disturbance inside the sphere.

While the fundamental equations for the æther are unaltered by the motion of electrified bodies, this is not the case with the equations for the moving matter itself.

As will be shown in Section 9, there is still considerable uncertainty as to what the true equations are. This difficulty does not, however, enter in the first order approximation.

If we refer to the equations for the æther in Section 5 and put k equal to zero, we see that the problem of a first order approximation there turns on a solution ϕ of the equations for a state of rest along with a solution depending on the initial field, the latter depending on the form of the equations.

Now in a similar way the disturbance in an insulating body will depend on a solution of the equations for the insulator at rest along with a solution depending on the form of the equations for a moving insulator and the initial field. Since, however, the initial field inside the sphere is zero the difficulty is removed, and we require only a solution of the equations for the insulator at rest, and these we know to be of the form given on p. 148 *supra*, with C replaced by C′, at all points where there is no charge. The units must be suitably chosen.

Hence at points outside the sphere the electric force is given by

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \frac{e}{r^3}(x, y, z) + \frac{\mathrm{C}}{r^3}(-1, 0, 0)(r^2\chi'' + r\chi' + \chi - e\xi/\mathrm{C})$$
$$+ \frac{\mathrm{C}x}{r^5}(x, y, z)(r^2\chi'' + 3r\chi' + 3\chi - 3e\xi/\mathrm{C}).$$

Inside the sphere we may have both converging and diverging disturbances, and hence we assume at all points inside

$$(X, Y, Z) = \frac{C'}{r^3}(-1, 0, 0)\{r^2\psi_1''(C't-r)+r^2\psi_2''(C't+r)+r(\psi_1'-\psi_2')+\psi_1+\psi_2\}$$
$$+\frac{C'x}{r^5}(x, y, z)\{r^2(\psi_1''+\psi_2'')+3r(\psi_1'-\psi_2')+3(\psi_1+\psi_2)\}.$$

Now the field must not become infinite at the origin, and this requires as a primary condition that

$$\psi_1(C't)+\psi_2(C't) = 0.$$

Neglecting squares of velocity, &c. $(X, Y, Z) = (X', Y', Z')$, and the tangential component of electric force must be continuous at $r = a$, and hence we get

$$C(a^2\chi''+a\chi'+\chi-e\xi/C) = C'\{a^2(\psi_1''+\psi_2'')+a(\psi_1'-\psi_2')+\psi_1+\psi_2\}.$$

Further, the surface density of electricity $\sigma = \frac{1}{4\pi}\frac{e}{a^2}$ and the difference of normal flux must equal $4\pi\sigma$.

We thus get the additional condition

$$C(a\chi'+\chi-e\xi/C) = KC'\{a(\psi_1'-\psi_2')+\psi_1+\psi_2\}.$$

The components of electric force at the surface are

Along normal outside $N_1 = \frac{e}{a^2}+\frac{2C.P_1}{a^3}(a\chi'+\chi-e\xi/C)$,

" tangent " $T_1 = -\frac{C.\sin\theta}{a^3}(a^2\chi''+a\chi'+\chi-e\xi/C)$,

" normal inside $N_2 = \frac{2C'.P_1}{a^3}\{a(\psi_1'-\psi_2')+\psi_1+\psi_2\}$,

" tangent " $T_2 = -\frac{C'.\sin\theta}{a^3}\{a^2(\psi_1''+\psi_2'')+a(\psi_1'-\psi_2')+\psi_1+\psi_2\}$.

Now the resultant tractions are

Along normal $\frac{1}{8\pi}\{N_1^2+(K-1)T_1^2-KN_2^2\}$,

" tangent $\frac{eT_1}{4\pi a^2}$.

Hence the force per unit area in the direction of x is

$$\frac{1}{8\pi}\{N_1^2+(K-1)T_1^2-KN_2^2\}\cos\theta+\frac{eT_1}{4\pi a^2}\sin\theta.$$

On integration for the whole sphere we get the comparatively simple value

$$-\tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi''.$$

Thus the equation of motion of the sphere is

$$m\ddot{\xi}+\tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi'' = \mathrm{F}.$$

Since the initial conditions are $\xi = 0$, $\dot{\xi} = 0$, when $t = 0$, and $\chi(-a) = \chi'(-a) = 0$, we get the integral

$$m\xi+\tfrac{2}{3}\frac{e}{a\mathrm{C}}\chi = \tfrac{1}{2}\mathrm{F}t^2.$$

Hence, eliminating ξ, we get the equations

$$\mathrm{C}\left\{a^2\chi''+a\chi'+\left(1+\frac{m'}{m}\right)\chi-\tfrac{1}{2}\frac{e\mathrm{F}t^2}{m\mathrm{C}}\right\} = \mathrm{C}'\{a^2(\psi_1''+\psi_2'')+a(\psi_1'-\psi_2')+\psi_1+\psi_2\},$$

$$\mathrm{C}\left\{a\chi'+\left(1+\frac{m'}{m}\right)\chi-\tfrac{1}{2}\frac{e\mathrm{F}t^2}{m\mathrm{C}}\right\} = \mathrm{KC}'\{a(\psi_1'-\psi_2')+\psi_1+\psi_2\}.$$

Thus, as a particular solution, we get

$$\chi(\mathrm{C}t-a) = \tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\mathrm{C}^3}\left[\mathrm{C}^2t^2-\frac{2ma\mathrm{C}t}{(m+m')}-2a^2\left\{\frac{mm'}{(m+m')^2}+\frac{m}{(\mathrm{K}-1)(m+m')}\right\}\right].$$

In addition, we must have vibratory terms in order to secure the satisfaction of the initial conditions. For this purpose we assume the forms

$$\chi(\mathrm{C}t-r) = \mathrm{A}e^{-\lambda(\mathrm{C}t-r+a)/a},$$

$$\psi_1(\mathrm{C}'t-r) = \mathrm{B}e^{-\mathrm{K}^{1/2}\lambda(\mathrm{C}'t-r+a)/a},$$

$$\psi_2(\mathrm{C}'t+r) = -\mathrm{B}e^{-\mathrm{K}^{1/2}\lambda(\mathrm{C}'t+r+a)/a}.$$

These satisfy the condition

$$\psi_1(\mathrm{C}'t)+\psi_2(\mathrm{C}'t) = 0.$$

Substituting, we obtain the equations

$$\mathrm{C}\left(\lambda^2-\lambda+1+\frac{m'}{m}\right)\mathrm{A} = \mathrm{C}'\{\mathrm{K}\lambda^2-\mathrm{K}^{1/2}\lambda+1-(1+\mathrm{K}^{1/2}\lambda+\mathrm{K}\lambda^2)e^{-2\mathrm{K}^{1/2}\lambda}\}\mathrm{B},$$

$$\mathrm{C}\left(1+\frac{m'}{m}-\lambda\right)\mathrm{A} = \mathrm{KC}'\{1-\mathrm{K}^{1/2}\lambda-(1+\mathrm{K}^{1/2}\lambda)e^{-2\mathrm{K}^{1/2}\lambda}\}\mathrm{B}.$$

The values of λ are thus the roots of the equation

$$e^{-2\mathrm{K}^{1/2}\lambda}=\frac{(\mathrm{K}-1)\left(1+\frac{m'}{m}\right)-(\mathrm{K}-1)\left(1+\mathrm{K}^{1/2}+\mathrm{K}^{1/2}\frac{m'}{m}\right)\lambda+\left(\mathrm{K}-1-\mathrm{K}^{1/2}\frac{m'}{m}\right)\mathrm{K}^{1/2}\lambda^2-\mathrm{K}(\mathrm{K}^{1/2}-1)\lambda^3}{(\mathrm{K}-1)\left(1+\frac{m'}{m}\right)+(\mathrm{K}-1)\left(\mathrm{K}^{1/2}-1+\mathrm{K}^{1/2}\frac{m'}{m}\right)\lambda-\left(\mathrm{K}-1+\mathrm{K}^{1/2}\frac{m'}{m}\right)\mathrm{K}^{1/2}\lambda^2+\mathrm{K}(\mathrm{K}^{1/2}+1)\lambda^3}.$$

Inspection shows that there can be no purely imaginary root, and further that there must be an even number of positive roots, if any.

Again, for even moderate values of K the roots will be nearly those of

$$(\mathrm{K}-1)\left(1+\frac{m'}{m}\right)-(\mathrm{K}-1)\left(1+\mathrm{K}^{1/2}+\mathrm{K}^{1/2}\frac{m'}{m}\right)\lambda+\left(\mathrm{K}-1-\mathrm{K}^{1/2}\frac{m'}{m}\right)\mathrm{K}^{1/2}\lambda^2-\mathrm{K}\,(\mathrm{K}^{1/2}-1)\,\lambda^3=\epsilon,$$

provided the root makes ϵ very small.

Now this equation suggests a single positive root, but examination of the initial form of the function shows that this root cannot differ from 0. As regards the remaining two roots of the cubic, it is readily shown that real positive values of λ are impossible, while real negative values cannot make ϵ small. Hence we have a pair of complex roots, the real part being positive.

For still greater values of K, the equation approximates to the form for a conductor, namely,

$$1+\frac{m'}{m}-\lambda+\lambda^2=0.$$

There is, of course, also the possibility of other complex roots, just as in the dynamical case of an elastic sphere vibrating in air. Without entering on the determination of these, it seems reasonable to expect that the vibrations are of a damped harmonic type and rapidly subside. They will be considered in Section 11. When this condition has been secured, we get the solution in the form

$$\chi(\mathrm{C}t-r)=\tfrac{1}{2}\frac{e\mathrm{F}}{(m+m')\mathrm{C}^3}\left[(\mathrm{C}t-r+a)^2-\frac{2ma(\mathrm{C}t-r+a)}{(m+m')}-2a^2\left\{\frac{mm'}{(m+m')^2}+\frac{m}{(\mathrm{K}-1)(m+m')}\right\}\right],$$

and hence

$$\xi=\tfrac{1}{2}\frac{\mathrm{F}}{(m+m')}\left[t^2+\frac{2m'at}{(m+m')\,\mathrm{C}}+2a^2\left\{\frac{m'^2}{(m+m')^2}+\frac{m'}{(\mathrm{K}-1)\,(m+m')}\right\}\right].$$

The result is very similar to that for a conducting sphere, differing only in the contribution to apparent initial displacement. We therefore conclude that an insulating sphere and a conducting sphere of equal radius and with equal surface charge possess equal electric inertia for slow speeds.

9. *Fundamental Equations for a Moving Dielectric.*—It would clearly be of considerable value if we could determine the accelerated motion of an insulating body at any speed, in the way that we have been able to determine it for a conductor.

As has been already mentioned in the preceding section, there is as yet no great degree of certainty as to the fundamental equations.

In addition to conforming to the ascertained fundamental laws and the laws for the æther as a special case, the equations must explain FRESNEL'S assumption as to the velocity of radiation in a moving body.

Two systems of equations may be proposed :—

First for axes fixed to the æther we may assume the system

$$\mathrm{C}'\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\ \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\ \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)=\left\{\frac{\partial}{\partial t}+\frac{2u\,(\mathrm{K}-1)}{\mathrm{K}}\frac{\partial}{\partial x}\right\}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z})$$

$$\mathrm{C}'\left(\frac{\partial\mathrm{Y}}{\partial z}-\frac{\partial\mathrm{Z}}{\partial y},\ \frac{\partial\mathrm{Z}}{\partial x}-\frac{\partial\mathrm{X}}{\partial z},\ \frac{\partial\mathrm{X}}{\partial y}-\frac{\partial\mathrm{Y}}{\partial x}\right)=\frac{\partial}{\partial t}(\alpha,\ \beta,\ \gamma)$$

for a body moving parallel to x with velocity u.

These give for the velocity of propagation of radiation the value

$$\pm\left\{\mathrm{C}'^2+\frac{u^2\,(\mathrm{K}-1)^2}{\mathrm{K}^2}\right\}^{1/2}+\frac{u\,(\mathrm{K}-1)}{\mathrm{K}},$$

which agrees with FRESNEL'S assumption as far as the first power of u.

Further, interpreting α, β, γ as magnetic and X, Y, Z as electric force, the equations contain FARADAY'S law. The convection current due to material polarisation, viz.,

$$2u\,\frac{(\mathrm{K}-1)}{\mathrm{K}}\frac{\partial}{\partial x}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z}),$$

is, however, difficult to explain on account of the factor 2.

Second, and again for fixed axes, we may assume the system

$$\mathrm{C}'\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z},\ \frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x},\ \frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)=\left\{\frac{\partial}{\partial t}+\frac{u\,(\mathrm{K}-1)}{\mathrm{K}}\frac{\partial}{\partial x}\right\}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z})$$

$$\mathrm{C}'\left(\frac{\partial\mathrm{Y}}{\partial z}-\frac{\partial\mathrm{Z}}{\partial y},\ \frac{\partial\mathrm{Z}}{\partial x}-\frac{\partial\mathrm{X}}{\partial z},\ \frac{\partial\mathrm{X}}{\partial y}-\frac{\partial\mathrm{Y}}{\partial x}\right)=\left\{\frac{\partial}{\partial t}+\frac{u\,(\mathrm{K}-1)}{\mathrm{K}}\frac{\partial}{\partial x}\right\}(\alpha,\ \beta,\ \gamma).$$

These give for the velocity of radiation

$$\pm\mathrm{C}'+\frac{u\,(\mathrm{K}-1)}{\mathrm{K}}$$

again in agreement with FRESNEL'S assumption.

In this system, which possesses the great advantage of symmetry, we may interpret α, β, γ as magnetic force. In order to reconcile the equations with FARADAY'S law we require to distinguish (X, Y, Z) as the æthereal electric force and

$$\left\{\mathrm{X},\ \ \mathrm{Y}+\frac{u\,(\mathrm{K}-1)}{\mathrm{C}'\mathrm{K}}\gamma,\ \ \mathrm{Z}-\frac{u\,(\mathrm{K}-1)}{\mathrm{C}'\mathrm{K}}\beta\right\}$$

as the total electric force. The convection current due to material polarisation is now

$$\frac{u\,(\mathrm{K}-1)}{\mathrm{K}}\frac{\partial}{\partial x}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z})$$

and presents no difficulty of interpretation.

A similar argument suggests the interpretation of X, Y, Z as electric force and a distinction between æthereal magnetic force α, β, γ and total magnetic force

$$\left\{\alpha, \quad \beta - \frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\mathrm{Z}, \quad \gamma = \frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\mathrm{Y}\right\}.$$

The existence of the additional electric force

$$\left\{0, \quad \frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\gamma, \quad -\frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\beta\right\}$$

is supported by WILSON'S ('Phil. Trans.,' A, vol. 204, p. 121) experiment of rotating an insulating cylinder in a longitudinal magnetic field. It would be equally important to test experimentally the existence of the additional magnetic force

$$\left\{0, \quad -\frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\mathrm{Z}, \quad \frac{u\,(\mathrm{K}-1)}{\mathrm{C'K}}\,\mathrm{Y}\right\}$$

by rotating the insulating cylinder in a longitudinal electric field. RÖNTGEN ('Ann. der Physik,' vol. 35, p. 264, 1888) has detected the existence of this force, but the effect was too small to be measured.

This second system is, I think, intrinsically involved in LORENTZ' and LARMOR'S equations, although not explicitly put in this symmetrical form, as far as I can find.

If this system could be established we could proceed to a higher degree of approximation in the problem of motion of an insulator, and the similarity of the equations to those for the æther shows that no greater analytical difficulty would arise.

We must still, however, remember the probable fact that K itself will be modified by higher order even powers of the velocity.

Thus until the accuracy of these equations or modifications of them is established beyond reasonable possibility of doubt, it would be a little absurd to apply them to the motion of an insulator for velocities comparable with that of radiation, and this consideration prevents me from attempting the solution of a problem which is clearly soluble from an analytical point of view by a method similar to that used for a perfect conductor.

Although we cannot therefore proceed to the general problem of a moving insulator at high speeds, we may show that if the dielectric ratio K is very large, the electric inertia will be very nearly the same as for a perfect conductor. Since there is continuity of normal flux of disturbed electric force at the surface, the functions which determine the disturbance inside the sphere are of order $1/\mathrm{K}$ as compared with those which determine the outside field. Hence the tangential component of electric force inside, and therefore also outside, is very nearly zero. Thus, since the equations for the æther are not modified by the motion of the sphere, the equation of motion and the surface forces outside differ by terms of order $1/\mathrm{K}$ from those for a perfect conductor. If this argument is valid, the assumption of perfect conduction, or of a high value of K for the charged particle, would equally well explain KAUFMANN's results, and give the same value for the electric inertia without limitation as to speed.

10. *Vibration of a Charged Conducting Sphere under a Periodic Force.*—While the examination of rectilinear motion is important from the general standpoint of electro dynamics, the problem of motion under periodic forces is no less important for optical theory.

Using the same notation as in Section 3, we shall assume the accelerating force to be purely mechanical and equal to F cos nt. In this case the sphere may be supposed never to move far from its original position, so that the approximation to the first order remains valid for an unlimited time.

The surface condition is, as before,

$$a^2\chi'' + a\chi' + \chi - e\xi/\mathrm{C} = 0,$$

and the equation of motion is

$$m\ddot{\xi} + \tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi'' = \mathrm{F}\cos nt,$$

the integral of which is

$$m\xi + \tfrac{2}{3}\frac{e}{a\mathrm{C}}\chi = \frac{\mathrm{F}}{n^2}(1-\cos nt).$$

Hence, if

$$m' = \tfrac{2}{3}e^2/a\mathrm{C}^2,$$

$$a^2\chi'' + a\chi' + \left(1 + \frac{m'}{m}\right)\chi = \frac{e\mathrm{F}\,(1-\cos nt)}{mn^2\mathrm{C}}.$$

Rapidly damped harmonic vibrations are initially produced, and when these subside we shall have only the residual effect of these with the particular integral.

Thus

$$\chi(\mathrm{C}t-r) = \frac{e\mathrm{F}}{(m+m')\,n^2\mathrm{C}} - \frac{e\mathrm{F}}{mn^2\mathrm{C}}\,\frac{\left\{\left(1+\frac{m'}{m}-\frac{a^2n^2}{\mathrm{C}^2}\right)\cos\frac{n(\mathrm{C}t-r+a)}{\mathrm{C}} + \frac{an}{\mathrm{C}}\sin\frac{n(\mathrm{C}t-r+a)}{\mathrm{C}}\right\}}{\left(1+\frac{m'}{m}-\frac{a^2n^2}{\mathrm{C}^2}\right)^2 + \frac{a^2n^2}{\mathrm{C}^2}},\text{*}$$

$$\xi = \frac{\mathrm{F}}{(m+m')\,n^2} - \frac{\mathrm{F}\cos nt}{mn^2} + \frac{m'\mathrm{F}}{m^2n^2}\,\frac{\left\{\left(1+\frac{m'}{m}-\frac{a^2n^2}{\mathrm{C}^2}\right)\cos nt + \frac{an}{\mathrm{C}}\sin nt\right\}}{\left(1+\frac{m'}{m}-\frac{a^2n^2}{\mathrm{C}^2}\right)^2 + \frac{a^2n^2}{\mathrm{C}^2}}.\text{*}$$

It is possible to interpret this as a solution of the equation

$$\mathrm{M}\ddot{\xi} - k\dddot{\xi} = \mathrm{F}\cos nt,\text{*}$$

where

$$\mathrm{M} = \frac{(m+m') - (m+m')\frac{a^2n^2}{\mathrm{C}^2} + m\frac{a^4n^4}{\mathrm{C}^4}}{1 - \frac{a^2n^2}{\mathrm{C}^2} + \frac{a^4n^4}{\mathrm{C}^4}}$$

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[*] In the corresponding expressions at 'Roy. Soc. Proc.,' A, vol. 77, 1906, p. 272, note the error of sign. Also in M the third term of numerator should be +.

and

$$k = \frac{m'\dfrac{a}{\mathrm{C}}}{1-\dfrac{a^2n^2}{\mathrm{C}^2}+\dfrac{a^4n^4}{\mathrm{C}^4}}.$$

If squares and higher powers of $\frac{an}{\mathrm{C}}$ may be neglected, we get $\mathrm{M} = m+m'$ and $k = \frac{2}{3}e^2/\mathrm{C}^3$, and this agrees exactly with the equation proposed by LORENTZ for the motion of an electron.

If we calculate the rate of radiation by means of the dissipation function it is found that the mean rate of radiation is

$$\tfrac{1}{3}\frac{e^2\mathrm{F}^2}{\mathrm{C}^3m^2}\Big/\left\{\left(1+\frac{m'}{m}-\frac{a^2n^2}{\mathrm{C}^2}\right)^2+\frac{a^2n^2}{\mathrm{C}^2}\right\},$$

and this is also the result obtained by calculating the Poynting flux. We thus obtain complete confirmation of LARMOR's result for a vibrating electron.

11. *First Order Vibrations of an Insulating Charged Sphere.*—From Section 8 it appears that the free vibrations of the first order are determined by the equations

$$\begin{aligned}
\mathrm{C}\{a^2\chi''+a\chi'+\chi-e\xi/\mathrm{C}\} &= \mathrm{C}'\{a^2(\psi_1''+\psi_2'')+a(\psi_1'-\psi_2')+\psi_1+\psi_2\},\\
\mathrm{C}\{a\chi'+\chi-e\xi/\mathrm{C}\} &= \mathrm{KC}'\{a(\psi_1'-\psi_2')+\psi_1+\psi_2\},\\
m\ddot{\xi}+\tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi'' &= 0,\\
\psi_1(\mathrm{C}'t)+\psi_2(\mathrm{C}'t) &= 0.
\end{aligned}$$

The assumption of a form $\chi(\mathrm{C}t-r) = \mathrm{A}e^{-\lambda(\mathrm{C}t-r+a)/a}$, with appropriate forms for ψ_1 and ψ_2, led to the equation for λ, viz.,

$${}^{-2\mathrm{K}^{1/2}\lambda} = \frac{(\mathrm{K}-1)\left(1+\dfrac{m'}{m}\right)-(\mathrm{K}-1)\left\{1+\mathrm{K}^{1/2}\left(1+\dfrac{m'}{m}\right)\right\}\lambda+\left(\mathrm{K}-1-\mathrm{K}^{1/2}\dfrac{m'}{m}\right)\mathrm{K}^{1/2}\lambda^2-\mathrm{K}(\mathrm{K}^{1/2}-1)\lambda^3}{(\mathrm{K}-1)\left(1+\dfrac{m'}{m}\right)+(\mathrm{K}-1)\left\{\mathrm{K}^{1/2}\left(1+\dfrac{m'}{m}\right)-1\right\}\lambda-\left(\mathrm{K}-1+\mathrm{K}^{1/2}\dfrac{m'}{m}\right)\mathrm{K}^{1/2}\lambda^2+\mathrm{K}(\mathrm{K}^{1/2}+1)\lambda^3}.$$

This equation may also be put in the form

$$\tanh \mathrm{K}^{1/2}\lambda = \mathrm{K}^{1/2}\lambda\left\{1+\frac{\mathrm{K}\lambda^2\left(1+\dfrac{m'}{m}-\lambda\right)}{(\mathrm{K}-1)\left(1+\dfrac{m'}{m}-\lambda\right)-\mathrm{K}\dfrac{m'}{m}\lambda+\mathrm{K}\lambda^3}\right\}.$$

If the sphere has no resultant charge or is held fixed, the equation becomes

$$\tanh \mathrm{K}^{1/2}\lambda = \mathrm{K}^{1/2}\lambda\left\{1+\frac{\mathrm{K}\lambda^2(1-\lambda)}{(\mathrm{K}-1)(1-\lambda)+\mathrm{K}\lambda^3}\right\}.$$

This equation is, allowing for the differences of notation, the same as that obtained by LAMB ('Camb. Phil. Trans.,' STOKES' Commemoration Volume, 1899). In that paper the equation is discussed on the supposition that the sphere is of atomic dimensions, and that K, the dielectric ratio, is exceedingly great. Hence, assuming λ to be very small, we get the approximate equation

$$\tanh \mathrm{K}^{1/2}\lambda = \mathrm{K}^{1/2}\lambda.$$

In this way LAMB shows that there are a number of roots, the wave-lengths corresponding to which may be large multiples of the diameter of the sphere, and thus in the vicinity of the visible spectrum.

A further approximation gives the modulus of decay, and in this way we find that the first root is given by

$$\lambda = \pm i\frac{4{\cdot}493}{\mathrm{K}^{1/2}} + \frac{(4{\cdot}493)^4}{\mathrm{K}^5}.$$

For such values of K as are contemplated by LAMB (10^6) the modulus of decay is exceedingly minute, thus indicating a high degree of persistence of the vibrations when once excited.

We have, however, seen in Section 8 that a pair of roots occurs in another way. When K is large and λ not small the period equation becomes approximately

$$\lambda^2-\lambda+1+\frac{m'}{m} = 0,$$

or

$$\lambda^2-\lambda+1 = 0$$

if the sphere is uncharged or fixed.

This gives

$$\lambda = \pm i\frac{\sqrt{3}}{2}+\tfrac{1}{2}.$$

Thus, in addition to the vibrations considered by LAMB, we have a vibration for which the wave-length is comparable with the diameter of the sphere, and of which the modulus of decay is very great.

This rapidly damped vibration corresponds to the vibration of a conductor.

This vibration has not been considered by LAMB in his paper; and it plays, as we shall show, a very important part in the optical behaviour of a sphere of atomic size with a large dielectric ratio.

For optical purposes it is necessary to determine the effect of a train of plane waves on the sphere, and this problem has been solved for a fixed sphere by LAMB (*loc. cit.*) and LOVE ('Proc. Lond. Math. Soc.,' vol. 30, 1899).

As is well known, the process consists in revolving the incident waves into terms proportional to spherical harmonics of different orders and finding the excited vibration which will satisfy the necessary surface conditions.

If we attempted to carry out this process rigorously for a charged sphere the problem would be very complex, because, in addition to the linear motion of the sphere, rotation would also be set up. Doubtless the problem is well worth investigation, but it is beyond the scope of the equations so far developed.

It appears, however, that for a wave-length of incident waves which is large in comparison with the radius of the sphere, by far the most important term in the incident waves is that corresponding to a spherical harmonic of the first order. This is the term which gives rise to linear motion of the sphere with associated first order vibrations. I therefore propose to limit the calculation to this order.

The equations at the beginning of this section have now to be modified by the introduction of the harmonic term due to the exciting waves, and we might then proceed to complete determination of ξ, χ, ψ_1 and ψ_2. We may, however, with advantage, simplify the matter at the outset by remembering that for such a high value of K as we contemplate, ψ_1 and ψ_2 are in general of order $1/\mathrm{K}^{1/2}$ as compared with χ and ξ.

Thus for a train of waves in the direction of z, for which the electrical force parallel to x is $\mathrm{F}e^{ik(\mathrm{C}t+z)}$, the equations for the first order vibrations are

$$m\ddot{\xi}+\tfrac{2}{3}\frac{e\mathrm{C}}{a}\chi'' = \frac{e\mathrm{F}\sin ka}{ka}\,e^{ik\mathrm{C}t},$$

$$\chi''+\frac{1}{a}\chi'+\frac{1}{a^2}\left(\chi-\frac{e\xi}{\mathrm{C}}\right) = -\frac{a\mathrm{F}}{\mathrm{C}}\,\tfrac{3}{2}\,\frac{\{(1-k^2a^2)\sin ka-ka\cos ka\}}{k^3a^3}\,e^{ik\mathrm{C}t}.$$

These equations are exact for a conductor and approximately true for an insulator, the terms neglected being of order $1/\mathrm{K}$.

Taking the real part of the solution, we get

$$\chi(\mathrm{C}t-r) = -\tfrac{3}{2}\,\frac{\mathrm{F}}{\mathrm{C}k^3}\,\frac{\left\{\left(1+\dfrac{m'}{m}-k^2a^2\right)\sin ka-ka\cos ka\right\}}{\left(1+\dfrac{m'}{m}-k^2a^2\right)^2+k^2a^2}$$
$$\times\left\{\left(1+\frac{m'}{m}-k^2a^2\right)\cos k(\mathrm{C}t-r+a)+ka\sin k(\mathrm{C}t-r+a)\right\},$$

$$\xi = -\frac{e\mathrm{F}\sin ka}{ma\mathrm{C}^2k^3}\cos k\mathrm{C}t-\tfrac{2}{3}\,\frac{e}{ma\mathrm{C}}\,\chi(\mathrm{C}t-a).$$

These give the forced part of the excited motion, and we should have to add terms depending on the free vibrations.

It is generally supposed that a vibratory motion of a charged sphere is attended by the emission of radiation. This is proved by application of the Poynting flux, and the field is supposed to be determined by a function χ, which is identified with $e\xi/\mathrm{C}$, while the exciting field is totally neglected. Now the exciting field must be included in the calculation, and our equations show that the relation between ξ, χ, and the exciting field is not so simple as that generally assumed.

We have shown in Section 3 that the motion here considered is associated with a dissipation function

$$\mathrm{D} = \tfrac{1}{3}\frac{\mathrm{C}}{a^2}\dot{\chi}\left(\dot{\chi} - \frac{e\dot{\xi}}{\mathrm{C}}\right).$$

It thus appears natural to suppose that the rate of radiation is 2D, and we have shown in Section 10 that this agrees with the calculation by means of POYNTING'S Theorem when the motive force is purely mechanical and the proper relation between χ and ξ is observed.

In the present case it appears that D may become negative, a result which can only be interpreted as meaning absorption and not emission of radiation. Although this result is somewhat novel, it is quite consistent with the common-sense view that there may be circumstances in which a vibrating particle absorbs radiation and others in which it emits radiation.

Substituting the values of $\dot{\chi}$ and $\dot{\xi}$ in D, and taking the mean value for a complete period, we find that the average rate of radiation is given by

$$\tfrac{3}{4}\frac{\mathrm{CF}^2}{a^2k^4}\frac{\left\{\left(1+\frac{m'}{m}-k^2a^2\right)\sin ka - ka\cos ka\right\}\left\{\left(1+\frac{m'}{m}-k^2a^2\right)\sin ka - \left(1+\frac{m'}{m}\right)ka\cos ka\right\}}{\left(1+\frac{m'}{m}-k^2a^2\right)^2+k^2a^2}.$$

Now the roots of the equations

$$\tan ka = ka\Big/\left(1+\frac{m'}{m}-k^2a^2\right)$$

and

$$\tan ka = \left(1+\frac{m'}{m}\right)ka\Big/\left(1+\frac{m'}{m}-k^2a^2\right),$$

which are real, are in general different. We therefore have regions for which there is radiation, separated by regions for which there is absorption.

The above expression for the radiation is true only if ka is a small quantity, and this must be observed in discussing the application of the expression to actual fact.

The positively charged particles associated with the electric discharge appear to be of atomic size and to have a ratio of mass to charge of the order of the electro-chemical equivalent of hydrogen. For such a particle the ratio of electric mass to ordinary mass is comparatively small.

If we take provisionally

$$\frac{e}{\mathrm{C}} = 1\times 10^{-20}, \qquad m = 1\times 10^{-24}, \qquad a = 1\times 10^{-8},$$

we get

$$\frac{m'}{m} = {\cdot}67 \times 10^{-8}.$$

Now the equation

$$\tan ka = ka \bigg/ \left(1 + \frac{m'}{m} - k^2a^2\right)$$

has a root $ka = 0$, and if $\frac{m'}{m}$ is small, a root given approximately by $k^2a^2 = \frac{3}{2}\frac{m'}{m}$. The other roots make ka finite, and are of no further concern here.

Similarly, the equation

$$\tan ka = \left(1 + \frac{m'}{m}\right) ka \bigg/ \left(1 + \frac{m'}{m} - k^2a^2\right)$$

has a root $ka = 0$, and for $\frac{m'}{m}$ small the other roots make ka finite.

Under these conditions the rate of radiation is given approximately by

$$\tfrac{1}{2} Ca^2F^2 \left(\tfrac{2}{3} k^2a^2 - \frac{m'}{m}\right).$$

From $ka = 0$ to $ka = \left(\frac{3}{2}\frac{m'}{m}\right)^{1/2}$ the expression is negative and above $ka = \left(\frac{3}{2}\frac{m'}{m}\right)^{1/2}$ it is positive. We thus conclude that the particle absorbs radiation and sends it out after in a conical beam for wave-lengths from infinity to a certain value, and for shorter wave-lengths would emit radiation, which is the normal condition.

It is to be understood that the exciting source is a train of plane electric waves, for with a purely mechanical exciting force there is always emission, according to the result in Section 10.

With the preceding value of $\frac{m'}{m}$ we find that the critical point, at which the change from absorption to emission takes place, is given by $ka = 10^{-4}$. This corresponds to light of wave-length about ten times that for sodium light.

The value could readily be brought into the vicinity of the visible spectrum by taking a particle made up of a group.

The true mass of such a composite particle would be proportional to the number of components, while the electric mass would be proportional to the square, and thus m' could be increased.

We have already referred to LAMB's conclusion that, if the dielectric ratio is of order 10^6, the free periods come in the vicinity of the visible spectrum. We have also noted the free period given by $ka = \frac{1}{2}\sqrt{3}$, which is necessarily in the ultra-violet, and for which the agitation of the particle and, consequently, the excitation of the other free vibrations must be abnormally increased.

If the exciting period does not exactly coincide with a free period, we may use the approximate equations to show that there is emission of radiation in those free periods.

The relation between the parts of χ and ξ depending on a free period is

$$m\xi+\tfrac{2}{3}\frac{e\chi}{a\mathrm{C}}=0,$$

so that

$$\chi-\frac{e\xi}{\mathrm{C}}=\left(1+\frac{m'}{m}\right)\chi.$$

Thus the approximate dissipation function is

$$\tfrac{1}{3}\frac{\mathrm{C}}{a^2}\left(1+\frac{m'}{m}\right)\dot{\chi}^2,$$

and this is necessarily positive. If, however, the exciting period exactly coincides with a free period, the approximation is invalid, and we cannot draw this conclusion.

Turning next to the case of negatively charged particles, these carry the same electric charge as the positively charged particles, but have a much smaller true mass. As a consequence m'/m for these is no longer small, but of finite order. Our deduction from KAUFMANN'S experiments gave m' of about the same order ot magnitude as m. This gives a value for $a = 1{\cdot}6\times10^{-13}$ which is much less than the atomic radius.

Consequently a value of $\mathrm{K} = 10^{16}$ would be required to give a free period in the visible spectrum.

When m'/m is finite, the roots of

$$\tan ka = ka\Big/\left(1+\frac{m'}{m}-k^2a^2\right)$$

make ka finite and do not further concern us.

The roots of

$$\tan ka = \left(1+\frac{m'}{m}\right)ka\Big/\left(1+\frac{m'}{m}-k^2a^2\right),$$

also, in general, make ka finite. But if m'/m is just less than 2, there is a possible root which makes ka small, given approximately by

$$k^2a^2\left(\tfrac{2}{5}-\tfrac{1}{10}\frac{m'}{m}\right)=2\left(1-\frac{m'}{2m}\right).$$

The rate of radiation is approximately

$$\tfrac{1}{4}\mathrm{C}a^2\mathrm{F}^2\left\{\frac{m'}{m}-\left(\tfrac{2}{3}+\tfrac{1}{6}\frac{m'}{m}\right)k^2a^2\right\}\left\{2\left(\frac{m'}{2m}-1\right)+k^2a^2\left(\tfrac{2}{5}-\tfrac{1}{10}\frac{m'}{m}\right)\right\}.$$

Hence, if m'/m is greater than 2, there is emission from infinite wave-length to very far out in the ultra-violet.

If m'/m is less than 2, there is absorption from infinite wave-length to a certain wave-length which depends on the closeness of m'/m to 2. Unless m'/m is very nearly 2, it will be in the ultra-violet.

With the present estimates it appears that in general the radiation in the visible spectrum from a single negative particle is but $\frac{1}{10,000}$th of the radiation from a single positive particle, and if m'/m is very nearly 2, the fraction will be still smaller.

We have thus proved important optical differences between positively and negatively charged particles. The results appear to have some application in the theory of excited luminosity, but as experimental knowledge of this subject is making rapid progress, this does not appear a suitable occasion on which to discuss a possible application of the results of this section.

12. *Slow Rotation of a Charged Sphere.*—The linear motion of a charged sphere is, as we have seen, attended by a disturbance in the surrounding æther. This disturbance gives a reaction on a moving sphere which is a single force. The fundamental forms which we found it necessary to assume are sometimes spoken of as disturbances of the first type. They might with propriety be called disturbances of electric type.

The skew-symmetry of the equations for the æther suggests that we should interchange the expressions for electric and magnetic force with the requisite change of sign. The disturbances represented by such forms are spoken of as of the second type, and might be called of magnetic type.

It at once appears that disturbances of this second or magnetic type give a reaction on a charged sphere which is not a single force, but is a couple tending to rotate the sphere. We thus have the means of investigating the problem of an accelerated motion of rotation of the sphere, similar in general character to the method developed for dealing with accelerated linear motion. This problem, which appears to have attracted some attention (LORENTZ, 'Enc. d. Math. Wiss.,' Vol. V., 2, Part 1, pp. 182–194, and others) in modern electron theory, is of considerable importance in general electrodynamics, and clearly falls within the scope of the present essay.

Taking, therefore, the case of a uniformly charged sphere, and assuming a disturbance of the second type depending on a first order zonal harmonic, the state of the æther outside the sphere is given by

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \frac{e}{r^3}(x, y, z) + \frac{\mathrm{C}}{r^3}(0, z, -y)(r\chi''+\chi'),$$

$$(\alpha, \beta, \gamma) = \frac{\mathrm{C}}{r^3}(-1, 0, 0)(r^2\chi''+r\chi'+\chi) + \frac{\mathrm{C}}{r^5}(x^2, xy, xz)(r^2\chi''+3r\chi'+3\chi).$$

We may first observe that if the sphere is a perfect conductor, we require that the tangential component of electric force should vanish when $r = a$.

Hence

$$a\chi''(\mathrm{C}t-a)+\chi'(\mathrm{C}t-a) = 0.$$

Thus the disturbance is not of a vibratory character, but is of a purely exponential type.

It may be observed that since the tangential component of electric force must vanish at the surface of a conductor, no couple on the conductor can arise from this cause. There remains the contribution to the tangential component of electrodynamic force on account of the magnetic field. If this arises from the motion contemplated it will give terms of the second order in the velocity of rotation, and may safely be neglected. Thus to this order there can be no æther reaction on the conductor due to its rotation. This does not imply independence of the rotation on the whole magnetic field. If, for instance, the external force is due to a rotating magnetic field, surface currents will be set up and a couple produced which will set the sphere in rotation.

If, however, the sphere is an insulator, the tangential component of electric force is no longer zero, and it will appear that there is a resultant couple on the sphere.

The equations for the inside of the sphere are, of course, altered by the assumed rotation; but just as in the problem of linear motion, we require a solution which is small of the first order in the angular velocity of the sphere, and hence neglecting terms involving squares of the angular velocity, the equations for insulators at rest suffice.

Inside the sphere we have both diverging and converging disturbances represented by $\psi_1(C't-r)$ and $\psi_2(C't+r)$ respectively.

Thus for the field inside we assume

$$(X, Y, Z) = \frac{C'}{r^3}(0, z, -y)\{r(\psi_1''-\psi_2'')+\psi_1'+\psi_2'\}$$

$$K^{-1/2}(\alpha, \beta, \gamma) = \frac{C'}{r^3}(-1, 0, 0)\{r^2(\psi_1''+\psi_2'')+r(\psi_1'-\psi_2')+\psi_1+\psi_2\}$$
$$+\frac{C'}{r^5}(x^2, xy, xz)\{r^2(\psi_1''+\psi_2'')+3r(\psi_1'+\psi_2')+3(\psi_1+\psi_2)\}.$$

In these equations $K = C^2/C'^2$ and the factor $K^{-1/2}$ is introduced in the form for magnetic force inside in order to make the units of measurement the same outside and inside.

Further, χ, ψ_1, and ψ_2 are supposed to be small quantities proportional to the angular velocity ω, and squares of ω are neglected.

Since the field must be finite at the origin, we must have

$$\psi_1(C't)+\psi_2(C't) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The normal component of magnetic force is continuous at $r = a$. Thus

$$C(a\chi'+\chi) = K^{1/2}C'\{a(\psi_1'-\psi_2')+\psi_1+\psi_2\},$$

or

$$a\chi'+\chi = a(\psi_1'-\psi_2')+\psi_1+\psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Inspection shows that this secures the continuity of the tangential component of electric force at $r = a$.

The discontinuity of tangential components of magnetic force determines the surface current which is due to the rotation of the uniform surface charge $\sigma = e/4\pi a^2$ rotating with the sphere.

Thus the angular velocity round the x axis from OY to OZ is given by ω where

$$ea^2\omega = \mathrm{C}^2\left[a^2\chi''+a\chi'+\chi-\{a^2(\psi_1''+\psi_2'')+a(\psi_1'-\psi_2')+\psi_1+\psi_2\}\right],$$

or in virtue of (2),

$$e\omega = \mathrm{C}^2(\chi''-\psi_1''-\psi_2'') \quad \ldots\ldots\ldots \quad (3).$$

The tangential component of electric force from OY to OZ is

$$\mathrm{T} = -\frac{\mathrm{C}}{a^2}\sin\theta\,(a\chi''+\chi').$$

Thus the total moment of the couple on the sphere, which is $\int\sigma\mathrm{T}a\sin\theta\,d\mathrm{S}$ reduces to

$$-\tfrac{2}{3}\frac{e\mathrm{C}}{a}(a\chi''+\chi'),$$

and this is the æther reaction on the sphere.

The motion contemplated may be originated by an extraneous electromagnetic disturbance, in which case our conditional equations would have to be slightly modified. But for clearness it is desirable to suppose that the sphere is acted on by a purely mechanical couple of magnitude L. Hence, if the sphere is uniform and of mass m, the equation of motion is

$$m\tfrac{2}{5}a^2\dot{\omega} = \mathrm{L}-\tfrac{2}{3}\frac{e\mathrm{C}}{a}(a\chi''+\chi'),$$

or

$$m\tfrac{2}{5}a^2\dot{\omega}+\tfrac{2}{3}\frac{e\mathrm{C}}{a}(a\chi''+\chi') = \mathrm{L} \quad \ldots\ldots\ldots \quad (4).$$

The equations (1) to (4) determine the motion, and may clearly be presented in a purely dynamical form.

The case of uniform rotation presents no special interest here. Passing to the case of a constant external couple, it appears that a uniformly accelerated rotation is possible. Without entering on algebraic details, we find that

$$\omega = \tfrac{5}{2}\frac{\mathrm{L}t}{(m+\frac{5}{6}m')\,a^2},$$

$$\chi(\mathrm{C}t-r) = \tfrac{5}{6}\frac{e\mathrm{L}}{\mathrm{C}^3(m+\frac{5}{6}m')}(\mathrm{C}t-r),$$

$$\psi_1(\mathrm{C}'t-r) = \tfrac{5}{96}\frac{\mathrm{K}^{1/2}e\mathrm{L}}{a^3\mathrm{C}^3(m+\frac{5}{6}m')}(\mathrm{C}'t-r)^4,$$

$$\psi_2(\mathrm{C}'t+r) = -\tfrac{5}{96}\frac{\mathrm{K}^{1/2}e\mathrm{L}}{a^3\mathrm{C}^3(m+\frac{5}{6}m')}(\mathrm{C}'t+r)^4,$$

constitutes a solution which satisfies all the equations.

Thus a uniformly accelerated rotation is possible, and the reaction of the medium is equivalent to an increase of the effective inertia of the sphere $= \frac{5}{6}m'$, m' being the electric inertia for linear motion at slow speeds.

The moment of inertia of the mass $\frac{5}{6}m'$ uniformly distributed is $\frac{5}{6}m' \times \frac{2}{5}a^2$ or $\frac{1}{3}m'a^2$, and is the same as that of a thin shell of mass $\frac{1}{2}m'$ and radius a. Thus for both linear and rotary uniform acceleration the dynamical effect of the æther is represented by the addition of a uniform thin shell of mass $\frac{1}{2}m'$ on the surface of the sphere, along with a particle of mass $\frac{1}{3}m'$ at the centre of the sphere.

As in the case of linear motion, this state is not attained without the production of initial vibratory disturbance which may be supposed to decay rapidly. The determination of these turns on the occurrence of free modes of motion, the arbitrary constants being determined by the initial conditions.

The integral of (4), when $\mathrm{L} = 0$, is

$$m\tfrac{2}{5}a^2\omega + \tfrac{2}{3}\frac{e}{a}(a\chi' + \chi) = 0,$$

and thus the free vibrations are determined by

$$a\chi' + \chi = a(\psi_1' - \psi_2') + \psi_1 + \psi_2,$$

$$a^2\chi'' + \tfrac{5}{2}\frac{m'}{m}(a\chi' + \chi) = a^2(\psi_1'' + \psi_2''),$$

$$\psi_1(\mathrm{C}'t) + \psi_2(\mathrm{C}'t) = 0.$$

Assuming the forms

$$\chi(\mathrm{C}t - r) = \mathrm{A}e^{-\lambda(\mathrm{C}t - r + a)/a},$$

$$\psi_1(\mathrm{C}'t - r) = \mathrm{B}e^{-\mathrm{K}^{1/2}\lambda(\mathrm{C}'t - r + a)/a},$$

$$\psi_2(\mathrm{C}'t + r) = -\mathrm{B}e^{-\mathrm{K}^{1/2}\lambda(\mathrm{C}'t - r + a)/a},$$

we get

$$\mathrm{A}(1 - \lambda) = \mathrm{B}\{(1 - \mathrm{K}^{1/2}\lambda) - (1 + \mathrm{K}^{1/2}\lambda)e^{-2\mathrm{K}^{1/2}\lambda}\},$$

$$\mathrm{A}\left\{(1 - \lambda)\tfrac{5}{2}\frac{m'}{m} + \lambda^2\right\} = \mathrm{BK}\lambda^2(1 - e^{-2\mathrm{K}^{1/2}\lambda}).$$

Thus λ is determined as a root of

$$e^{-2\mathrm{K}^{1/2}\lambda} = \frac{(1 - \mathrm{K}^{1/2}\lambda)\left\{(1 - \lambda)\tfrac{5}{2}\frac{m'}{m} + \lambda^2\right\} - (1 - \lambda)\mathrm{K}\lambda^2}{(1 + \mathrm{K}^{1/2}\lambda)\left\{(1 - \lambda)\tfrac{5}{2}\frac{m'}{m} + \lambda^2\right\} - (1 - \lambda)\mathrm{K}\lambda^2}$$

or

$$\tanh \mathrm{K}^{1/2}\lambda = \frac{\mathrm{K}^{1/2}\lambda\left\{(1 - \lambda)\tfrac{5}{2}\frac{m'}{m} + \lambda^2\right\}}{(1 - \lambda)\tfrac{5}{2}\frac{m'}{m} + \lambda^2 - (1 - \lambda)\mathrm{K}\lambda^2}.$$

The first of these forms shows that there is a real root for λ which will not differ much from unity when K is large. It corresponds to a purely exponential disturbance.

Again, if K is very great and λ very small, but so that $K^{1/2}\lambda$ is finite, the second form is approximately

$$\tanh K^{1/2}\lambda = K^{1/2}\lambda\tfrac{5}{2}\frac{m'}{m}\Big/\left(\tfrac{5}{2}\frac{m'}{m}-K\lambda^2\right).$$

Hence

$$\lambda = \pm iz,$$

where

$$\tan K^{1/2}z = K^{1/2}z\tfrac{5}{2}\frac{m'}{m}\Big/\left(\tfrac{5}{2}\frac{m'}{m}+Kz^2\right),$$

the roots of which in general make $K^{1/2}z$ finite. These correspond to a vibratory disturbance, and a nearer approximation gives the damping coefficient.

The problem of rotation of a vibratory character may also be treated by aid of the equations. In dealing with the problem of linear vibration, approximate treatment for large values of K was possible since ψ_1 and ψ_2 were small compared with χ.

With rotary vibration ψ_1, ψ_2, and χ are of the same order, and approximate treatment is no longer possible. I do not propose to give the results of this problem, as they are somewhat cumbrous, and do not appear to present any important optical features. Such a conclusion may be expected from the investigations of RAYLEIGH ('Phil. Mag.,' p. 379, 1899), LAMB, and LOVE (*loc. cit.*), which show that the radiation for disturbance of the second type is insignificant compared with radiation for disturbance of the first type. In this connexion it is interesting to observe the nature of the mode of linking of the sphere to the æther in the two types. In both we have the slowly damped vibrations which may be of wave-length large compared with the diameter of the sphere if K is great. The main link is, however, through the rapidly damped vibration of wave-length comparable with the diameter, in the case of a linear vibration, and the purely exponential disturbance in the case of rotary vibration.

The considerations in Section 9 prevent us from attempting to extend the results to the case of a high speed of rotation.

The independence of small linear motion and small rotary motion will be apparent from the method of examining the two cases, and we are thus able to present in a purely dynamical form the equations of motion of a sphere in general, provided the velocities are small compared with that of radiation.

The disturbance of exponential type, which occurs in the problem just treated, arises with all disturbances of magnetic type associated with zonal harmonics of odd order. It also occurs with all disturbances of electric type associated with zonal harmonics of even order. Pure damped harmonic vibrations, on the other hand, occur with disturbances of magnetic type associated with even order zonal harmonics, and disturbances of electric type associated with odd order zonal harmonics.

Changes of the linear motion of a charged sphere give rise to disturbances of electric type. Those depending on odd order harmonics are damped harmonic waves, while those depending on even order harmonics are a mixture of exponential and damped harmonic waves.

Similarly, changes of rotary motion of a charged sphere give rise to disturbances of magnetic type. Those depending on even order harmonics are damped harmonic waves, while those depending on odd order harmonics are a mixture of exponential and damped harmonic waves. That depending on the first order harmonic is, however, a purely exponential wave.

13. *Induced Electrification and Electric Vibrations on a Conducting Sphere moving with Uniform Speed.*—When a fixed spherical conductor is under the influence of an electrical field, the problem of finding the induced potential is, as is well known, a comparatively simple matter, an inducing potential involving the same spherical harmonic and no other.

If, however, a spherical conductor is constrained to move uniformly in a straight line in a specified electrical field, the problem is more difficult. An inducing normal force involving a given spherical harmonic no longer gives rise to an induced surface density involving only the same spherical harmonic, but terms involving other spherical harmonics may also arise.

The form of solution for a given inducing field depends on what we take as the proper boundary condition. As has been shown, we may take the condition (1) that the tangential component of electric force (X, Y, Z) should vanish at the surface of the sphere, or the condition (2) that the tangential component of (X′, Y′, Z′) should vanish at the surface of the sphere.

I propose now to give the solutions for the case of a spherical conductor constrained to move in the direction of x with a uniform velocity kC in a field of uniform force F, which is parallel to the direction of motion. With condition (2) the problem may be stated as follows:—

Determine a function, ϕ, which satisfies

$$(1-k^2)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0,$$

so that the tangential component of

$$(\mathrm{X}', \mathrm{Y}', \mathrm{Z}')=(1-k^2)\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

at $r=a$ shall be equal to the tangential component of

$$(1-k^2)\,\mathrm{F}\,(-1, 0, 0).$$

If, as before,

$$\frac{x^2}{\cosh^2\eta}+\frac{(1-k^2)(y^2+z^2)}{\sinh^2\eta}=k^2a^2,$$

we find that

$$\phi = \mathrm{A}x\left\{\log\coth\tfrac{1}{2}\eta - 1/\cosh\eta\right\}$$

where

$$\mathrm{A} = \mathrm{F}\Big/\left\{k - \tfrac{1}{2}\log\frac{(1+k)}{(1-k)}\right\}$$

is the solution.

With condition (1) the statement of the problem is:—Determine a function, ϕ, which satisfies

$$(1-k^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

so that the tangential component of

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \left\{(1-k^2)\frac{\partial\phi}{\partial x}, \quad \frac{\partial\phi}{\partial y}, \quad \frac{\partial\phi}{\partial z}\right\}$$

at $r = a$ shall be equal to the tangential component of

$$\mathrm{F}(-1, 0, 0).$$

If

$$\rho^2 = \frac{x^2}{1-k^2} + y^2 + z^2$$

$$\rho_1^{\,2} = \frac{(x-ka)^2}{1-k^2} + y^2 + z^2$$

$$\rho_2^{\,2} = \frac{(x+ka)^2}{1-k^2} + y^2 + z^2,$$

the solution is

$$\phi = \mathrm{A}x\left\{\log\coth\tfrac{1}{2}\eta - 1/\cosh\eta\right\} + \tfrac{1}{2}\,\frac{k^2a^2\mathrm{A}}{(1-k^2)^{1/2}}\,\frac{1}{\rho}\log\frac{\{(\rho+\rho_1)^2 - k^2a^2/(1-k^2)\}}{\{(\rho+\rho_2)^2 - k^2a^2/(1-k^2)\}}$$

where

$$\mathrm{A} = \mathrm{F}\Big/\left\{k - \frac{(1-k^2)}{2}\log\frac{(1+k)}{(1-k)}\right\}.$$

The surface density of electricity is given by

$$4\pi\sigma = \frac{\mathrm{A}ak^3x}{(1-k^2)\,\rho^2} - \tfrac{1}{2}\,\frac{k^2a^3\mathrm{A}}{(1-k^2)^{1/2}\,\rho^3}\log\frac{\{(\rho+\rho_1)^2 - k^2a^2/(1-k^2)\}}{\{(\rho+\rho_2)^2 - k^2a^2/(1-k^2)\}},$$

the appropriate values of ρ, ρ_1, and ρ_2 at the surface $r = a$ being substituted.

We have seen in Section 3 that the production of a uniformly accelerated slow motion is established by the aid of a rapidly damped harmonic train of waves and the period equation has been found.

Further, in Sections 5 and 6, where no limitation as to speed is made, we have indicated that the production of a uniformly accelerated change of the motion is established in a similar way. It was assumed that the initial vibrations set up were rapidly damped, but the equations given are not sufficient to give the period equation.

It is therefore desirable to discuss this point more fully, and it will be sufficient for our present purpose to consider the sphere as constrained to move uniformly in a straight line with velocity kC. We may state our problem as the determination of the period equation for first order vibrations on the sphere, such as would be excited by the production of a uniform field of electric force.

When the sphere is at rest the vibrations depending on zonal harmonics of different orders are quite independent of each other; but when the sphere is moving this is no longer the case. In particular, the vibrations excited in establishing a uniform field no longer depend solely on a zonal harmonic of the first order, but involve an infinite series of zonal harmonics, as might be expected from our examination of the steady distribution produced by a uniform field.

In general, the components of electric force are

$$\mathrm{X} = -\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2}, \qquad \mathrm{Y} = \frac{\partial^2\phi}{\partial y\,\partial x}, \qquad \mathrm{Z} = \frac{\partial^2\phi}{\partial z\,\partial x},$$

where

$$\left(\frac{\partial}{\partial t} - k\mathrm{C}\frac{\partial}{\partial x}\right)^2 \phi = \mathrm{C}^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right).$$

Since the field is to be symmetrical about the axis of x we may write

$$\mathrm{X} = -\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi\frac{\partial\phi}{\partial\varpi}, \qquad \mathrm{Y} = \frac{y}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial x}, \qquad \mathrm{Z} = \frac{z}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial x},$$

where

$$\left(\frac{\partial}{\partial t} - k\mathrm{C}\frac{\partial}{\partial x}\right)^2 \phi = \mathrm{C}^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial\phi}{\partial\varpi}\right).$$

If we make LORENTZ' transformation

$$t' = t - \frac{k}{1-k^2}\frac{x}{\mathrm{C}},$$

we get

$$\mathrm{X} = -\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial\phi}{\partial\varpi}, \quad \mathrm{Y} = \frac{y}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial x} - \frac{k}{(1-k^2)\mathrm{C}}\frac{y}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial t'}, \quad \mathrm{Z} = \frac{z}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial x} - \frac{k}{(1-k^2)\mathrm{C}}\frac{z}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial t'},$$

and

$$\frac{1}{(1-k^2)}\frac{\partial^2\phi}{\partial t'^2} = \mathrm{C}^2\left\{(1-k^2)\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial\phi}{\partial\varpi}\right\}.$$

We now assume condition (1), that the tangential component of X, Y, Z at $r = a$ should vanish.

Thus

$$\frac{k}{(1-k^2)\mathrm{C}}\frac{x}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial t'} - \frac{x}{\varpi}\frac{\partial^2\phi}{\partial\varpi\,\partial x} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial\phi}{\partial\varpi} = 0$$

when $r = a$.

This may be written

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left\{\frac{k}{(1-k^2)}\frac{x}{\mathrm{C}}\frac{\partial\phi}{\partial t'} - x\frac{\partial\phi}{\partial x} - \varpi\frac{\partial\phi}{\partial\varpi}\right\} = 0$$

when $r = a$, or

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left\{\frac{k}{(1-k^2)}\frac{x}{\mathrm{C}}\frac{\partial \phi}{\partial t'} - r\frac{\partial \phi}{\partial r}\right\}$$

when $r = a$.

In performing the partial differentiations, ϕ must first be expressed as a function of independent variables r and $\mu = \frac{x}{r}$, and after differentiating with respect to r, the functions must be expressed in terms of independent variables x and ϖ, and the differentiation with respect to ϖ performed.

Let

$$\phi \propto e^{(1-k^2)^{1/2}\mathrm{C}\theta t'}, \quad \text{and put} \quad \lambda = \frac{k}{(1-k^2)^{1/2}}.$$

The surface condition at $r = a$ may now be written

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left\{\lambda\theta x\phi - r\frac{\partial \phi}{\partial r}\right\} = 0.$$

The simplest form for ϕ is

$$\phi_0 = \frac{e^{-\theta\rho}}{\rho} \quad \text{where} \quad \rho^2 = \frac{x^2}{1-k^2} + y^2 + z^2 = r^2 + \lambda^2 x^2,$$

and derived forms are obtained by successive differentiations with respect to x.

We shall now approximate by neglecting terms involving higher powers of λ than λ^2.

Let

$$\phi = \mathrm{A}_0\phi_0 + \mathrm{A}_1\frac{\partial \phi_0}{\partial x} + \mathrm{A}_2\frac{\partial^2 \phi_0}{\partial x^2}$$

where

A_1 is of order λ,

A_2 is of order λ^2,

and higher terms are neglected.

Now

$$\phi_0 = \psi_0 + \Sigma_1^\infty \frac{\lambda^{2n} x^{2n}}{2^n\, n!}\psi_n$$

where

$$\psi_n = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{e^{-\theta r}}{r}.$$

Thus, as far as squares of λ, we have

$$\phi = \mathrm{A}_0(\psi_0 + \tfrac{1}{2}\lambda^2\mu^2 r^2\psi_1) + \mathrm{A}_1\mu r\psi_1 + \mathrm{A}_2(\psi_1 + \mu^2 r^2\psi_2),$$

where

$$\mu = x/r.$$

On performing the differentiations, as already explained, we find that the surface condition is

$$\begin{aligned}&-\mathrm{A}_0(\theta^2\psi_0 - \psi_1) - \mathrm{A}_2(\theta^2\psi_1 - 3\psi_2),\\ &+x\{\lambda\theta\mathrm{A}_0\psi_1 - \mathrm{A}_1(\theta^2\psi_1 - 2\psi_2)\},\\ &+x^2\{-\tfrac{1}{2}\lambda^2\mathrm{A}_0(\theta^2\psi_1 - \psi_2) + \lambda\theta\mathrm{A}_1\psi_2 - \mathrm{A}_2(\theta^2\psi_2 - 3\psi_3)\} = 0,\end{aligned}$$

when $r = a$.

Equating the coefficients of the various powers of x to zero, we get

$$-\tfrac{1}{2}\lambda^2 A_0 (\theta^2\psi_1 - \psi_2) + \lambda\theta A_1\psi_2 - A_2 (\theta^2\psi_2 - 3\psi_3) = 0,$$

$$\lambda\theta A_0\psi_1 - A_1 (\theta^2\psi_1 - 2\psi_2) = 0,$$

$$A_0 (\theta^2\psi_0 - \psi_1) + A_2 (\theta^2\psi_1 - 3\psi_2) = 0.$$

Eliminating A_0, A_1, and A_2, we get the period equation

$$\theta^2\psi_0 - \psi_1 = \frac{\lambda^2}{2} \frac{(\theta^4\psi_1{}^2 - 5\theta^2\psi_1\psi_2 + 2\psi_2{}^2)(\theta^2\psi_1 - 3\psi_2)}{(\theta^2\psi_2 - 3\psi_3)(\theta^2\psi_1 - 2\psi_2)}.$$

Now, when λ is entirely neglected, we get

$$\theta^2\psi_0 - \psi_1 = 0, \qquad \text{or} \qquad a^2\theta^2 + a\theta + 1 = 0,$$

which is our former period equation for a fixed sphere. The roots are

$$a\theta = -\tfrac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

We can now proceed to an approximation to $a\theta$ in λ^2 by putting

$$a\theta = -\tfrac{1}{2} \pm i\frac{\sqrt{3}}{2} + \zeta$$

on the left-hand side of the period equation, and substituting the value

$$a\theta = -\tfrac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

in the term on the right-hand side, which is of order λ^2. In this term we may use the approximate results

$$\psi_1 = \theta^2\psi_0, \qquad a^2\psi_2 = -2\theta^2\psi_0, \qquad \text{and} \qquad a^2\psi_3 = \theta^4\psi_0 + 10\frac{\theta^2}{a^2}\psi_0.$$

Hence we find

$$\zeta = \frac{\lambda^2}{260} \mp i\frac{\lambda^2 59\sqrt{3}}{780}.$$

Thus

$$a\theta = -\tfrac{1}{2}\left(1 - \frac{\lambda^2}{130}\right) \pm \frac{i\sqrt{3}}{2}\left(1 - \frac{\lambda^2 59}{390}\right).$$

We could, in a similar manner, proceed to the period equation for higher order vibrations, and to calculate higher approximations to the roots in powers of λ^2. The process would be tedious, and I have not as yet discovered any artifice for effecting the summation.

We have now proved that the time factor of first order vibrations is

$$e^{(1-k^2)^{1/2}C\theta t'},$$

where

$$a\theta = -\tfrac{1}{2}\left(1-\frac{\lambda^2}{130}\right) \pm \frac{i\sqrt{3}}{2}\left(1-\frac{\lambda^2 59}{390}\right),$$

$$\lambda^2 = k^2/(1-k^2),$$

and higher powers of λ^2 are neglected.

In a similar way condition (2) leads to

$$a\theta = -\tfrac{1}{2} \pm \frac{i\sqrt{3}}{2}\left(1+\frac{\lambda^2}{10}\right).$$

The result shows that both the frequency and the damping coefficient of the first order vibration diminish as the velocity of the sphere increases. The effect is clearly not considerable until the velocity is nearly that of radiation, and a higher degree of approximation for such a speed is necessary.

We have assumed that the sphere is constrained to move uniformly. If the sphere is uncharged no constraint is necessary, so that the solution applies directly to the case of an uncharged sphere. If the sphere is charged and unconstrained the equations are more complicated, so that I shall give only the result.

The time factor of the vibrations is

$$e^{(1-k^2)^{1/2}C\theta t'},$$

where

$$a\theta = -\tfrac{1}{2}\left(1+\frac{140}{507}\frac{m'}{m}\lambda^2-\frac{\lambda^2}{130}\right) \pm \tfrac{1}{2}\left(3+4\frac{m'}{m}\right)^{1/2}\left(1+\frac{1630}{1521}\frac{m'}{m}\lambda^2-\frac{59\lambda^2}{390}\right),$$

$$\lambda^2 = k^2/(1-k^2), \quad \text{and} \quad m' = \tfrac{2}{3}e^2/aC^2.$$

This approximation neglects squares of m'/m and higher powers of λ^2, and if we also agree to neglect products of m'/m and λ^2 we get, when condition (1) is used,

$$a\theta = -\tfrac{1}{2}\left(1-\frac{\lambda^2}{130}\right) \pm \tfrac{1}{2}\left(3+4\frac{m'}{m}\right)^{1/2}\left(1-\frac{59\lambda^2}{390}\right),$$

while condition (2) leads to

$$a\theta = -\tfrac{1}{2} \pm \tfrac{1}{2}\left(3+4\frac{m'}{m}\right)^{1/2}\left(1+\frac{\lambda^2}{10}\right).$$

So far as these calculations go they indicate the way in which the process of attempting to establish a uniform acceleration, at speeds nearly that of radiation, may fail. The damped harmonic train of waves may have such a small damping coefficient that it is no longer legitimate to neglect those waves, and a uniform acceleration cannot, in point of fact, be established.

We may further observe that an uncharged sphere of atomic size would, under ordinary conditions, give fundamental vibrations of frequency corresponding to extreme ultra-violet radiation, damped with exceedingly great rapidity. Our result proves that, at a speed approaching that of radiation, the fundamental vibration may be brought within the visible spectrum range, and at the same time the damping becomes relatively small.

A similar conclusion holds for an unconstrained charged sphere provided m'/m is not large.

These results are of significance in optical theory, and investigation of the effect of speed on the vibrations, carried to a higher degree of approximation, appears to be desirable.